Introduction to Proofs

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Review

What is covered in this course? - Basic tools and techniques - Precise and rigorous mathematical reasoning Why are proofs necessary? What is a (valid) proof in Mathematics? What details do you include or skip?

What is a proof?

 In Math, a proof is a step-by-step demonstration that a conclusion follows from some hypotheses.

In a each step use hypotheses, axioms, previously proven theorems, rules of inference, and logical equivalences such that the intermediate conclusion follows from previous step

Terminology

Theorem: A statement that can be proved to be true
Axiom: A statement which is given to be true
Lemma: A `pre-theorem' that is needed to prove a theorem

Corollary: A 'post-theorem' that follows from a theorem

Rules of Inference

- In order to infer new facts using facts we already have
- Rules of Inference are important in proofs, although they are sometimes used without being mentioned

Rules of Inference



 $H_1 \wedge H_2 \wedge ... \wedge H_n \rightarrow C$

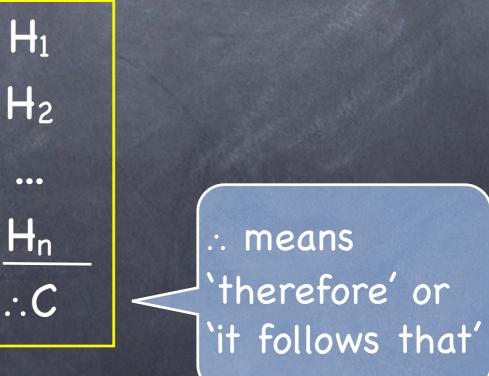
Hi : Hypotheses

Conclusion

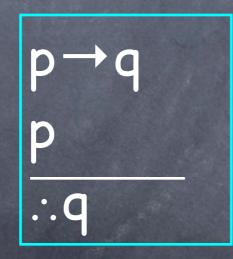
As a rule of inference they take the symbolic form:

...

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Modus Ponens (Law of Detachment)

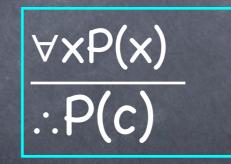


$$((p \rightarrow q) \land p) \rightarrow q$$

lautology

Universal Instantiation

 From ∀xP(x) is true we can infer that P(c) is true, where c is a particular member of the domain



Universal Generalization

 From P(c) is true for an arbitrary c in the domain, we can infer that ∀xP(x) is true.

More Rules of Inference

Read rules on page 66 and 70.

Onderstanding is required, memorization is not.

Types of Proofs

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Direct proof (including proof by cases) Proof by contraposition Proof by contradiction Proof by construction Proof by induction Other techniques

Indirect proof

Direct Proof

Leads from hypothesis to the conclusion

 \oslash How to prove $p \rightarrow q$?

- Assume p is true



These steps are constructed using:Rules of inference

- Axioms
- • Lemmas
 - Definitions
- Proven theorems

- q must be true

- Q.E.D. (used to signal the end of a proof) $\frac{12}{12}$

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Direct Proof (example) If n is an odd integer, then n² is odd.

Proof:

- Assume n is an odd integer
- By definition, n=2k+1 for some integer k
- $-n^2 = (2k+1)^2 = 4k^2+4k+1 = 2(2k^2+2k)+1$
- Let $m = 2k^2 + 2k$, $n^2 = 2m + 1$
- By definition, n^2 is odd

– Q.E.D.

Definition: n is an odd integer if n=2k+1 for some integer k

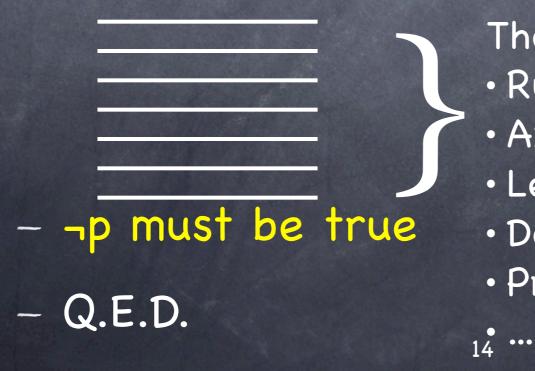
Proof by Contraposition

$p \rightarrow q \equiv \neg q \rightarrow \neg p$

Leads from the negation of conclusion to the negation of hypothesis

 \odot How to prove $p \rightarrow q$?

- Assume -q is true



These steps are constructed using:Rules of inference

- Axioms
- Lemmas
- Definitions
- Proven theorems

Proof by Contraposition (example)

If n^2 is an even integer, then n is even.

Proof (by contraposition):

- Assume n is an odd integer. Then n=2k+1 (k is integer)
- $-n^{2} = (2k+1)^{2} = 4k^{2}+4k+1 = 2(2k^{2}+2k)+1$
- Let integer $m = (2k^2+2k)$, then $n^2=2m+1$.
- So n² is odd.
- Q.E.D.

Proof by Contradiction

 $p \rightarrow q \equiv p \land \neg q \rightarrow FALSE$

Leads from the hypothesis and the negation of conclusion to a contradiction

 \oslash How to prove $p \rightarrow q$?

- Assume p and -q is true

- Contradiction!

– Q.E.D.

These steps are constructed using:

- Rules of inference
- Axioms
- Lemmas

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- Definitions
- Proven theorems

Proof by Contradiction (example)

 $\sqrt{2}$ is irrational.

Proof (by contradiction):

- Assume $\sqrt{2}$ is rational. Then $\sqrt{2}=a/b$ such that a and b have no common factors (definition)
- Squaring and transposing: $2=a^2/b^2$, $a^2=2b^2$.
- a² is even, so a is even (previous slide). i.e. ∃k a=2k
- $-a^2 = 4k^2 = 2b^2$, so $b^2 = 2k^2$
- b² is even, so b is even (previous slide). i.e. ∃m b=2m
- a and b have common factor 2 -- Contradiction!

Reading and Notes

Skim Sec 1.5, read Sec 1.6

Master the basic proof methods: direct proof, proof by contraposition, proof by contradiction

Recommended exercises: 1.5: 3,15,19,23; 1.6: 1,11,17