Nested Quantifiers

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Review

- Propositions, Logical operators -- basic
- Predicates, Quantifiers -- more powerful

Nested Quantifiers

- One quantifier can be placed within the scope of the other
- Allows simultaneous quantification of many variables

$$\boxed{P(x,y)}$$

$$\forall x \exists y (x+y=0)$$

$$\boxed{Q(x)}$$

$$\forall x \exists y (x+y=0)$$

$$\Downarrow$$

$$\forall x Q(x)$$

$$\Downarrow$$

$$Q(x):\ \exists y P(x,y)$$

$$\Downarrow$$

$$P(x,y):\ x+y=0$$

Nested Quantifiers (example)

- Show $\forall x \exists y(x+y=0)$ is true over the integers
  - Assume an arbitrary integer x
  - To show that there exists a y that satisfies the requirement of the predicate, choose $y = -x$. Clearly y is an integer, i.e. in the domain.
  - So $x + y = x + (-x) = 0$.
  - Since we assumed nothing about x (other than it is an integer), the argument holds for any integer x.
  - Therefore, the predicate is TRUE.

Order of Nested Quantifiers

- When there are only one kind of quantifiers (universal or existential) in a statement, then the change of order does not change the meaning of the statement:

$$\forall x \forall y P(x,y) \equiv \forall y \forall x P(x,y)$$

$$\exists x \exists y P(x,y) \equiv \exists y \exists x P(x,y)$$

Order of Nested Quantifiers -2

- When there are different quantifiers, order matters
- Consider the two propositions over the integer domain:

  $\forall x \exists y (x<y)$ v.s $\exists y \forall x (x<y)$

  - $\forall x \exists y (x<y)$: "there is no maximum integer"
  - $\exists y \forall x (x<y)$: "there is a maximum integer"
    - Not the same meaning at all!!

Quantifications of Two Variables

Statement	When true?	When false?
$\forall x \forall y P(x,y)$ $\forall y \forall x P(x,y)$	P(x,y) is true for every pair (x,y)	A pair (x,y) exists for which P(x,y) is false
$\forall x \exists y P(x,y)$	For every x, there is a y for which P(x,y) is true	There is an x such that P(x,y) is false for every y
$\exists x \forall y P(x,y)$	There is an x for which P(x,y) is true for every y	For every x, there is a y for which P(x,y) is false
$\exists x \exists y P(x,y)$ $\exists y \exists x P(x,y)$	There is a pair (x,y) for which P(x,y) is true	P(x,y) is false for all pairs (x,y)

Examples

What is the truth value of the following:

- $\exists x \forall y (x+y=0)$ domain: integers
- $\exists x \forall y (xy=0)$ domain: integers
- $\forall x \neq 0 \exists y (y=1/x)$ domain: real numbers
- $\forall x \forall y \exists z (z=(x+y)/2)$ domain: integers

Negation of Nested Quantifiers

- Review: Negation of quantification
  - $\neg\forall x P(x) \equiv \exists x \neg P(x)$
  - $\neg\exists x P(x) \equiv \forall x \neg P(x)$

Negation of Nested Quantifiers

Same rules as before

Ex 1: $\neg\forall x\exists y(x<y)$

$\equiv \exists x\neg\exists y(x<y)$

$\equiv \exists x\forall y\neg(x<y)$

$\equiv \exists x\forall y(x\geq y)$

Ex 2: $\neg\exists x\forall y(x+y=0)$

$\equiv \forall x\neg\forall y(x+y=0)$

$\equiv \forall x\exists y\neg(x+y=0)$

$\equiv \forall x\exists y(x+y\neq 0)$

English -> Logical Expression

- "The sum of two positive integers is always positive."
  - Rewrite in English using quantifiers and domain:

    "For every pair of integers, if both integers are positive, then the sum of them is positive."
  - Introduce variables

    "For integers x and y, if x and y are positive, then x +y is positive."
  - $\forall x \forall y((x>0) \land (y>0) \rightarrow (x+y>0))$ domain: integers

English -> Logical Expression (2)

- “No one has more than one best friend.”
  - Determine individual propositional function

    BF(x,y): y is the best friend of x

  - Rewrite in English using variables, BF, and quantifiers:

    “There does not exist one x, for whom there is a person y and a person z other than y such that BF (x,y) and BF(x,z) are both true.”

  - $\neg\exists x\exists y\exists z(BF(x,y)\land BF(x,z)\land(z\neq y))$ domain: all people

English -> Logical Expression (3)

- Transform the logical expression to a logical equivalent one

$\neg\exists x\exists y\exists z(BF(x,y)\wedge BF(x,z)\wedge(z\neq y))$ domain: all people

$\equiv \forall x\neg\exists y\exists z(BF(x,y)\wedge BF(x,z)\wedge(z\neq y))$

Negation of quantifiers

$\equiv \forall x\forall y\neg\exists z(BF(x,y)\wedge BF(x,z)\wedge(z\neq y))$

$\equiv \forall x\forall y\forall z\neg(BF(x,y)\wedge BF(x,z)\wedge(z\neq y))$

De Morgan's Laws

$\equiv \forall x\forall y\forall z(\neg BF(x,y)\vee\neg BF(x,z)\vee\neg(z\neq y))$

$\equiv \forall x\forall y\forall z(\neg BF(x,y)\vee\neg BF(x,z)\vee(z=y))$

Logical Expression -> English

- $\forall x(hasComputer(x) \vee \exists y(hasComputer(y) \wedge friends(x,y)))$

  Domain of x and y: all students

- "For every student x, x has a computer or there is a student y such that y has a computer and x and y are friends"

- "Every student has a computer or has a friend who has a computer"

Logical Expression -> English (2)

- $\exists x \forall y \forall z(friends(x,y) \wedge friends(x,z) \wedge (y \neq z) \rightarrow \neg friends(y,z))$

  Domain of x, y and z: all students

- "There is a student x such that for all students y and all students z, if x and y are friends, x and z are friend and z and y are not the same student, then y and z are not friend."

- "There is a student none of whose friends are also friends with each other."

Readings and Notes

- Read Sec 1.4
- Understand the order and scope of the quantification
- Practice translating between English and logical expressions
- Practice proving/showing the truth value of a logical expression
- Recommended exercises: 1,3,9,12,21,25,27,39