Predicates and Quantifiers

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Overview

- Review: Propositions, Logical operators, Logical equivalence
- Limitations of propositional logic:
  - Refer to (constant) objects
  - Also need to say:
    - objects have certain properties
    - objects relate to one another in certain ways
- Predicate Logic is more powerful

Predicate

- A predicate is a proposition that is a function of one or more variables.

$x > 3$ → $P(x)$

Variable

Predicate:
Property the variable can have

Name of the predicate

Variable

Predicate - 2

- A predicate is a proposition that is a function of one or more variables.

$x > y$ ⇒ $P(x,y)$

Variable

Predicate:
Relationship of the two variables

Name of the predicate

Variables

Predicate (Example) - 1

Positive(x): x>0

What are the truth values of P(4) and P(-2)?

Solution:

- x=4

  P(4): 4>0 --------True

- x=-2

  P(2): -2>0 --------False

Predicate (Example) - 5

Greater(x,y): x>y

What are the truth values of Greater(4,1) and Greater (2,2)?

Solution:

- x=4,y=1

  Greater(4,1): 4>1 -------True

- x=2,y=2

  Greater(2,2): 2>2 -------False

Quantifier

- By describing the range of the variable (AKA. binding) it becomes possible to determine the truth value of the predicate
- Two popular quantifiers
  - Universal: $\forall x P(x)$ - "P(x) is true for <u>all</u> x in the <u>domain</u>"
  - Existential: $\exists x P(x)$ - "P(x) is true for <u>some</u> x in the <u>domain</u>"

Universal Quantifier

- Universal quantifier $\forall$:
  - For all…; For every…; For each…; All of…; For arbitrary…
- Using universal quantifier (domain: real numbers)
  - $\forall x(x^2 \leq 0)$
  - $(\forall x > 1)(x^2 > x)$ - quantifier with restricted domain

Universal Quantifier -2

$\forall x P(x)$:

- When true?

  P(x) is true for every x in the domain

- When false?

  P(x) is not always true when x is in the domain (find a value of x that P(x) is false)

  Counterexample

Universal Quantifier Examples

Domain: real numbers

- P(x): $x^2 \geq 0$ Is $\forall x P(x)$ true?
  - $x^2 \geq 0$ is true for all real numbers, so $\forall x P(x)$ is true
- Q(x): $x^2 > x$ Is $\forall x Q(x)$ true?
  - Find a counterexample: when $x=0$, $Q(0)$: $0^2>0$ is false, so $\forall x P(x)$ is false

Existential Quantifier

- Existential quantifier $\exists$:
  - There exists…; There is…; For some…; For at least one…
- Using existential quantifier (domain: real numbers)
  - $\exists x(x>1)$
  - $\exists x(x=x+1)$

Existential Quantifier -2

$\exists x P(x)$:

- When true?

  There is an x in the domain for which P(x) is true (find a value of x that P(x) is true)

- When false?

  P(x) is false for every x in the domain

Existential Quantifier Examples

Domain: real numbers

- P(x): x>1 Is ∃xP(x) true?
  - Find an x such that P(x) is true: when x=100, P(x): 100>1 is true, so ∃xP(x) is true
- Q(x): x=x+1 Is ∃xQ(x) true?
  - Q(x) is false for all real numbers, so ∃xP(x) is false

A tricky example

- "Every CS student is smart"
  - $\forall x\ (CSStudent(x) \rightarrow Smart(x))$
- "Some CS student is smart"
  - $\exists x\ (CSStudent(x) \land Smart(x))$
- What is the difference of the following?
  - $\forall x\ (CSStudent(x) \land Smart(x))$
  - Why it cannot represent "Every CS student is smart"?

Scope of Quantifiers

- $\forall, \exists$ have higher precedence than operators from Propositional Logic
  - E.g. $\exists x P(x) \vee Q(x)$ is not logically equivalent to $\exists x (P(x) \vee Q(x))$

Operators	Precedence
$\forall$ $\exists$	0
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Logical Equivalence

Logically equivalent?

$\forall x(P(x) \land Q(x))$	Yes	$\forall x P(x) \land \forall x Q(x)$
$\forall x(P(x) \lor Q(x))$	No	$\forall x P(x) \lor \forall x Q(x)$
$\exists x(P(x) \land Q(x))$	No	$\exists x P(x) \land \exists x Q(x)$
$\exists x(P(x) \lor Q(x))$	Yes	$\exists x P(x) \lor \exists x Q(x)$

Negation of Quantifiers

- De Morgan's Laws
  - $\neg\forall x P(x) \equiv \exists x \neg P(x)$
  - $\neg\exists x P(x) \equiv \forall x \neg P(x)$
- Careful: The negation of "Every professor is good" is NOT "No professor is good"!!

Readings and Notes

- Read Section 1.3
- Understand the difference and relationship between propositions, predicates(functional propositions), and predicates with quantifications
- Practice translating English using predicate logic
- Recommended exercises: 13,21,23,25,33,39