Resolution and Refutation

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Overview

- Propositional Logic
  - Resolution
  - Refutation
- Predicate Logic
  - Substitution
  - Unification
  - Resolution
  - Refutation
  - Search space

[ref.: Nilsson- Chap.3]

[also Prof. Zbigniew Stachniak's notes]

Theorems from Logic

[from Mathematical Logic, George Tourlakis]

- Modus Ponens $A, A \rightarrow B \vdash B$
- Cut Rule $A \vee B, \neg A \vee C \vdash B \vee C$

$A, \neg A \vdash \bot$

- Transitivity of → $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$
- Proof by Contradiction $\Gamma \vdash A \;\; \textit{iff} \;\; \Gamma + \neg A \vdash \bot$

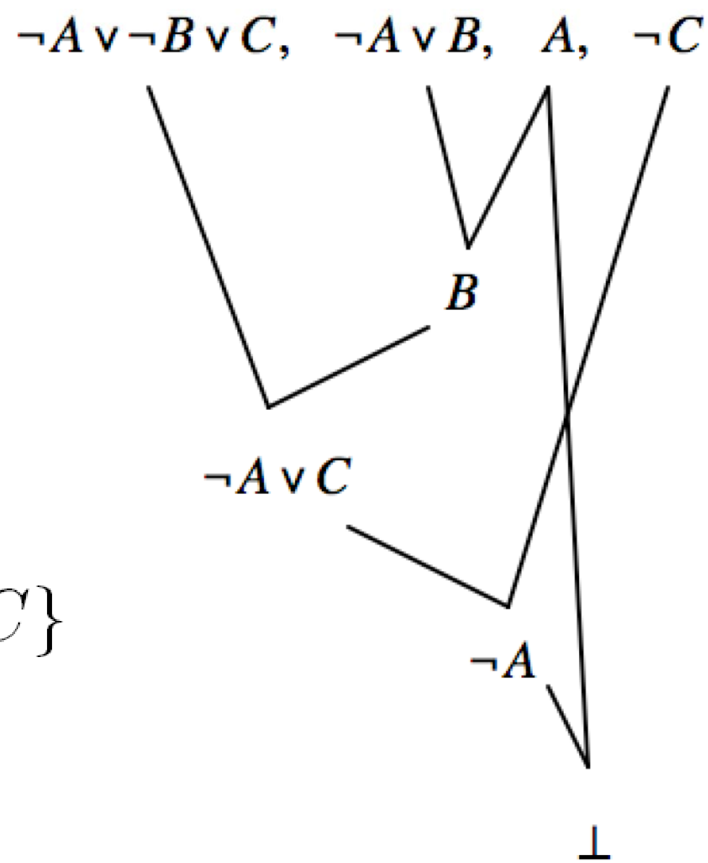
Resolution in Logic

- By A. Robinson (1965)
- Example: Prove

$$A \rightarrow (B \rightarrow C), A \rightarrow B, A \vdash C$$

- We need to show that the

$$\{\neg A \vee \neg B \vee C, \neg A \vee B, A, \neg C\}$$



Resolution in Logic Programming

- Program P (facts and rules in clause form)
- Goal G negated and added to program P
- To prove G, we need to show $P + \{\neg G\}$ is inconsistent

Complementary literals

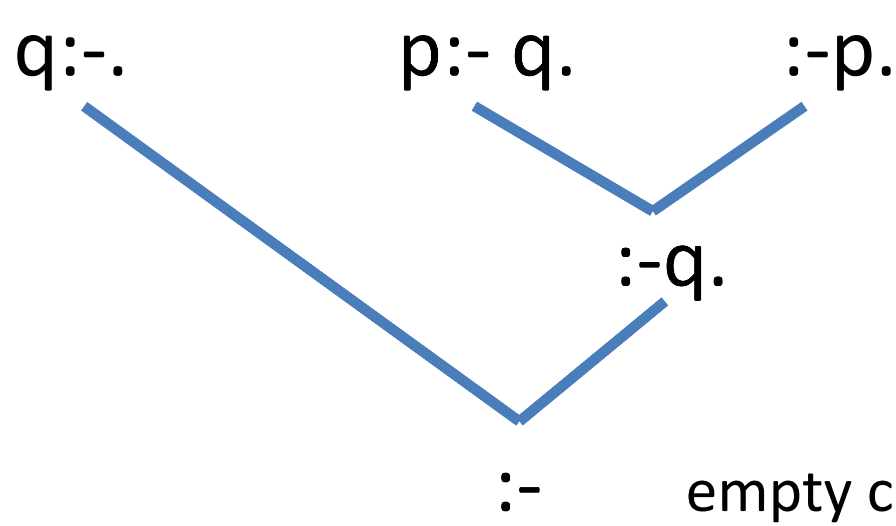
q, t :- p. , p :- s. ← **Parent clauses**

← **Resolving upon p, being on different sides of :-**

q, t :- s. ← **Resolvent**

Example (1)

- Program P={q:-. , p:-q.}
- Query :-p.
  - This is already the negated form of our goal!



empty clause, inconsistency
therefore p is satisfiable → true

Refutation

- When resolution is used to prove inconsistency, it is called refutation. (refute=disprove)
- The above binary tree, showing resolution and resulting in the empty clause, is called a refutation tree.
- NOTE: To avoid potential mistakes, DO NOT RESOLVE UPON MORE THAN ONE LITERAL SIMULTANEOUSLY.

Example (2)

A1. If Henry has two days off, then if the weather is bad, Henry is not fishing.

A2. if Henry is not fishing and is not drinking in a pub with his friends, then he is watching TV at home.

A3. If Henry is working, then he is neither drinking in a pub with his friends nor watching TV at home.

Q. If Henry is not watching TV at home and he has two days off, then he is drinking in a pub with his friends provided that the weather is bad.

Example (2) (cont.)

- From logical point of view, we want to prove Q, given A1, A2, A3. $\{A1, A2, A3\} \vdash Q$.
- By refutation principle, the consistency of

$$\mathbf{C} = \{A1,\ A2,\ A3\} \bigcup \{\neg Q\}$$

  is examined.
  - Step 1: Represent as propositional formulas
  - Step 2: Represent as clauses
  - Step 3: Determine the consistency of C
    - If C is consistent, answer NO (false)
    - If C is inconsistent, answer YES (true)

Example (2) (cont.)

A1. If Henry has two days off, then if the weather is bad, Henry is not fishing.

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Q. If Henry is not watching TV at home and he has two days off, then he is drinking in a pub with his friends provided that the weather is bad.

p: H has two days off
q: weather is bad
r: H is fishing
s: H is drinking in a pub with his friends
t: H is watching TV at home
u: H is working

A1. p ->(q ->~r)
A2. (~r & ~s) -> t
A3. u -> (~s & ~t)

Q. (~t & p) -> (q->s)

Example (2) (cont.)

- Conversion to clause form

$$A1: p \rightarrow (q \rightarrow \neg r) \Rightarrow \neg p \vee \neg q \vee \neg r \Rightarrow C_1 = \ :- p, q, r.$$

$$A2: (\neg r \wedge \neg s) \rightarrow t \Rightarrow \neg(\neg r \wedge \neg s) \vee t \Rightarrow r \vee s \vee t \Rightarrow C_2 = \ r, s, t :-.$$

$$A3: u \rightarrow (\neg s \wedge \neg t) \Rightarrow \neg u \vee (\neg s \wedge \neg t) \Rightarrow (\neg u \vee \neg s) \wedge (\neg u \vee \neg t)$$

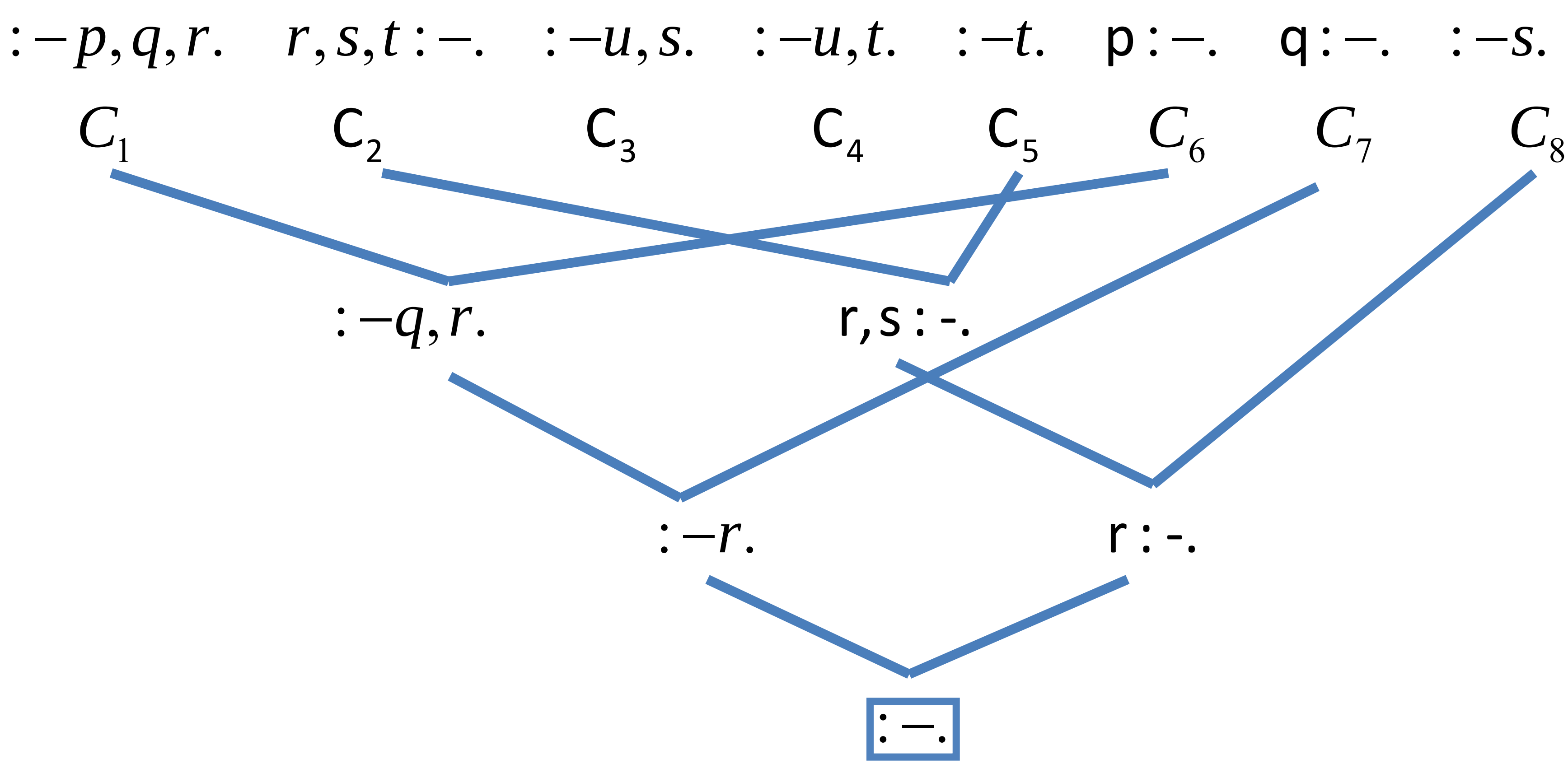
$$\Rightarrow \begin{cases} C_3 = \ :- u, s. \\ C_4 = \ :- u, t. \end{cases}$$

$$\neg Q: \neg\big((\neg t \wedge p) \rightarrow (q \rightarrow s)\big) \Rightarrow (\neg t \wedge p) \wedge \neg(\neg q \vee s) \Rightarrow \neg t \wedge p \wedge q \wedge \neg s$$

$$\Rightarrow \begin{cases} C_5 = \ :- t. \\ C_6 = \ \mathsf{p} :-. \\ C_7 = \ \mathsf{q} :-. \\ C_8 = \ :- s. \end{cases}$$

Example (2) (cont.)

- Determining the consistency of $\{C_1, C_2, \ldots, C_8\}$



Example (2) (cont.)

- C={C1, C2, ..., C8} is inconsistent (by resolution/ refutation)
- Therefore Q is provable (deducible)
- Answer: YES (true)
- This is how Prolog answers Queries. If the empty string is deduced, Prolog answers YES (or TRUE).

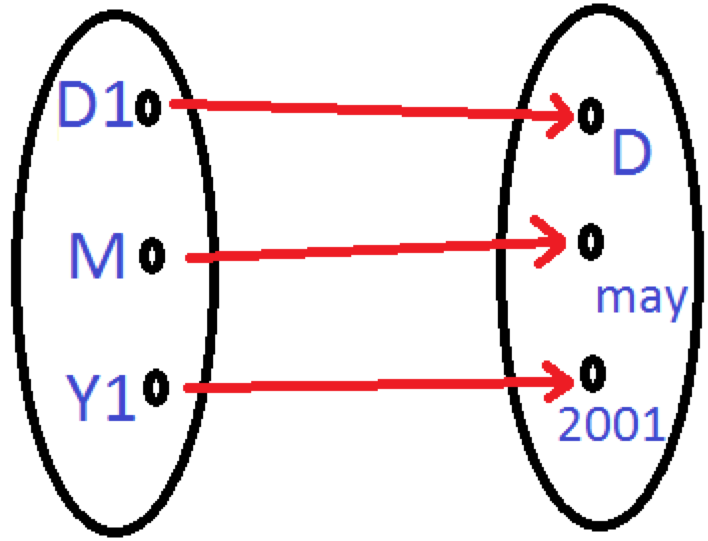
Resolution in Predicate Logic

- A literal in Predicate Logic (PL) is either
  - A positive literal in the form of $p(t_1,\ldots,t_k)$ where p is a predicate and t_i are terms
  - Or a negative literal in the form of $\neg p(t_1,\ldots,t_k)$
- Two clauses in PL can be resolved upon two complementary unifiable literals
- Two literals are unifiable if a substitution can make them identical.
- Example:
  - study_hard(X) and study_hard(john)
  - date(D, M, 2001) and date(D1, may, Y1)

Substitution

- Substitution: is a finite set of pairs of terms denoted as $[X_1/t_1, ..., X_n/t_n]$ where each t_i is a term and each X_i is a variable.

- Every variable is mapped to a term; if not explicitly mentioned, it maps to itself.



- For example:
  - date(D, M, 2001) and date(D1, may, Y1)

Applying substitution to literals

- Example:

  p(X, f(X, 2, Z), 5)

  e= [X/5, Z/h(a,2+X)]

  e(p(X, f(X, 2, Z), 5))= p(5, f(5, 2, h(a, 2+X)), 5)

- Note:
  - Simultaneous substitution
  - X in h(a,2+X) is not substituted

- Example:

  r(X, Y)

  e=[X/Y, Y/X]

  e(r(X, Y))= r(Y, X)

- Example:

  r(X, f(2, Y))

  e=[f(2, Y)/Z]

  illegal substitution- only variables can be substituted

Applying substitution to clauses

- Substitution of a clause is defined by applying substitution to each of its literals:

  $e(p :- q_1, ..., q_k.) = e(p) :- e(q_1), ..., e(q_k).$

- Example:

  C: pass_3401(X):- student(X, Y), study_hard(X).

  e=[X/ john, Y/ 3401]

  e(C)= pass_3401(john):- student(john, 3401), study_hard(john).

Unifier

- Let p_1 and p_2 be two literals and let *e* be a substitution. We call *e* a unifier of p_1 and p_2 if $e(p_1)=e(p_2)$.
- Two literals are unifiable if such a unifier exists.
- Example:

  date(D, M, 2001) and date(D1, may, Y1)

  e_1=[D/15, D1/ 15, M/may, Y1/2001]

  e_2=[D1/D, M/may, Y1/2001] ← A more general unifier

- A unifier *e* is said to be a most general unifier (mgu) of two literals/terms iff *e* is more general than any other unifier of the terms.

Unification

- Called <u>matching</u> in Prolog
- Rules for matching two terms S and T match [Bratko]:
  - If S and T are constants, then S and T match only if they are the same object.
  - If S is a variable (and T is anything), then they match and S is substituted by T (*instantiated to T*). Conversely, if T is a variable, then T is substituted by S.
  - If S and T are structures, then they match if
    - S and T have the same principal functor
    - All their corresponding components match

Unification vs. Matching

- Are p(X) and p(f(X)) unifiable?
  e=[X/f(X)]
  X=f(f(f(f(f(..... ?!

- This is not allowed in unification. Proper unification requires <u>occurs check</u>: a variable X can not be substituted by a term t if X occur in t.

- This is not done in Prolog's matching for efficiency reasons.
  - Therefore it is referred to as 'matching' in Prolog, and not 'unification'.

Examples

Are the following literals unifiable? What is their mgu?

1. triangle(point(1,2), X, point(2,4)) and triangle(A, point(5, Y), point(2, B))
   unifiable: mgu=[A/point(1,2), X/point(5,Y), B/4]

2. horizontal(point(1,X), Y) and vertical(Z,A)
   not unifiable: horizontal ≠ vertical

3. plus(2,2) and 4
   not unifiable

4. seg(point(1,2), point(3,4)) and seg(f(1,2), Y)
   not unifiable: point ≠ f

The resolution rule

- Given two clauses in the form:
  $A_0..A_i..A_m$:-$B_1...B_n$. *and* $C_1...C_k$:- $D_1..D_j..D_l$.

  If e is a unifier of A_i and D_j (i.e. $e(A_i)=e(D_j)$)

  Then the resolvent of the above two clauses is:

  $e(A_0)..\ e(A_{i-1})e(A_{i+1})..\ e(A_m)\ e(C_1)..e(C_k)$:-
  $e(B_1)..\ e(B_n)\ e(D_1)..\ e(D_{j-1})e(D_{j+1})..\ e(D_l)$.

- Example:
  C_1: p(f(1)):- r(X, Y), q(Y, Z).
  C_2: :- p(Y).
  Unifier of p(f(1)) and p(Y): e=[Y/f(1)]
  The resolvent of C_1 and C_2: :- r(X, f(1)), q(f(1), Z).

Example

[Nilsson]

C0: proud(X) :- parent(X, Y), newborn(Y).
C1: parent(X, Y) :- father (X, Y).
C2: parent(X, Y) :- mother(X, Y).
C3: father(adam, mary).
C4: newborn(mary).

G0: :- proud(Z).

Unifier of proud(..) in C0 and G0: e=[X/Z], resolvent:
G1: :- parent(Z,Y), newborn(Y).
Unifier of parent(..) in C1 and G1: e=[X/Z, Y/Y], resolvent:
G2: :- father(Z,Y), newborn(Y).
<u>To prevent mistakes, we rename the variables whenever we use a fresh copy of a clause.</u>

Example (cont.)

[Nilsson]

G0: :- proud(Z).

(copy of) C0: proud(X_1) :- parent(X_1, Y_1), newborn(Y_1).

Resolve with G0: e=[X_1/Z]

G1: :- parent(Z,Y_1), newborn(Y_1).

(copy of) C1: parent(X_2,Y_2) :- father (X_2, Y_2).

Resolve with G1: e=[X_2/Z, Y_2/Y_1]

G2: :- father(Z, Y_1), newborn(Y_1).

(copy of) C3: father(adam, mary).

Resolve with G2: e=[Z/adam, Y_1/mary]

G3: :-newborn(mary).

(copy of) C4: newborn(mary).

Resolve with G3: e=[]

G4: :-

Empty clause → answer to query: <u>true</u> and <u>Z=adam</u>

Example (cont.)

[Nilsson]

Just a different notation for :-

$$\leftarrow proud(Z).$$

$$proud(X_1) \leftarrow parent(X_1, Y_1), newborn(Y_1).$$

$$\leftarrow parent(Z, Y_1), newborn(Y_1).$$

$$parent(X_2, Y_2) \leftarrow father(X_2, Y_2).$$

$$\leftarrow father(Z, Y_1), newborn(Y_1).$$

$$father(adam, mary).$$

$$\leftarrow newborn(mary).$$

$$newborn(mary).$$

$$\square$$

Refutation Tree for G0

Linear Refutation

- We can resolve with different clauses and keep adding new clauses forever!
- To prevent this, Linear Refutation always starts with a goal (as the example showed previously).
- Prolog's computation rule:

  Always selects the leftmost subgoal, although logically there is no order for the subgoals.

  Example: When resolving G1: :- parent(Z,Y_1), newborn(Y_1)., parent(..) was selected to resolve upon.

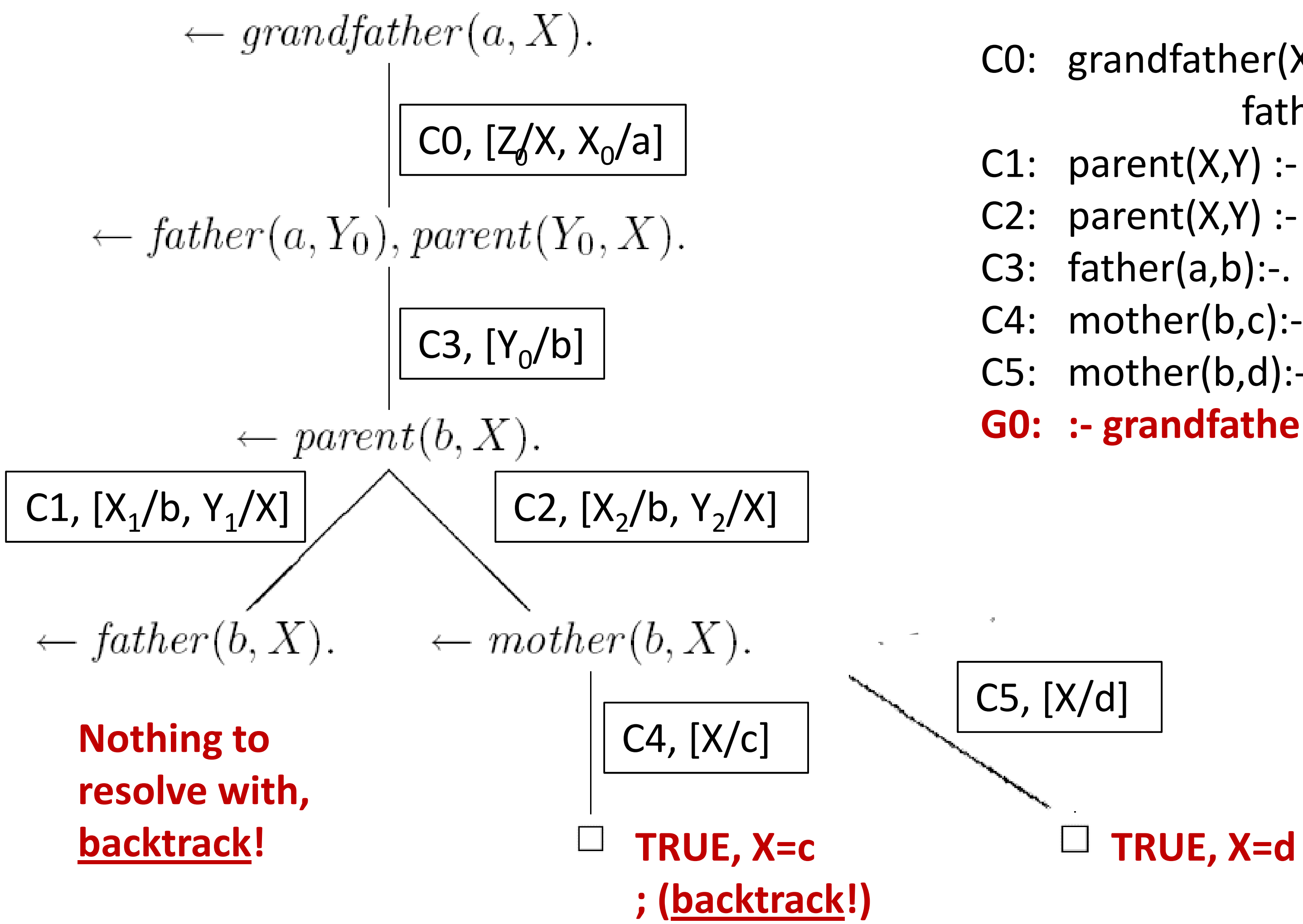
  Prolog also starts from the top of knowledge base and goes down the list of facts and rules.

Search Space

- Based on linear refutation and Prolog's computation rule, we know the search tree of Prolog.
- Search tree:
  The root in the search tree is the main goal G_0. A child node is a new goal G_i obtained through resolution. A link is labelled with the clause resolved with and the substitution.
- Example:

C0:	grandfather(X,Z) :- father(X,Y), parent(Y,Z).
C1:	parent(X,Y) :- father(X,Y).
C2:	parent(X,Y) :- mother(X,Y).
C3:	father(a,b):-.
C4:	mother(b,c):-.
C5:	mother(b,d):-.
G0:	**:- grandfather(a,X).**

Search Space (example)



C0: grandfather(X,Z) :- father(X,Y), parent(Y,Z).

C1: parent(X,Y) :- father(X,Y).

C2: parent(X,Y) :- mother(X,Y).

C3: father(a,b):-.

C4: mother(b,c):-.

C5: mother(b,d):-.

G0: :- grandfather(a,X).

Search Space

- What is the search strategy used by Prolog for searching the tree?

Depth First Search or Breadth First Search

Prolog uses DFS