Math/CS 1019
Practice Final

Name:

Student #:

1. **10 pts**

Consider the following algorithm:

Input: two positive integers a and b
$x \leftarrow 1$
for i from 1 to b
    $x \leftarrow xa$
Output: x

What does it do?

What is its running time (in big-O notation)? (Justify your answer)

2. **10 pts**

Prove that the set of multiples of five and the set of multiples of three have the same cardinality.

3. **20 pts**

Let S be the subset of the set of ordered pairs of integers defined recursively by

Basis Step: $(0, 0) \in S$

Recursive Step: If $(a, b) \in S$ then $(a + 2, b + 3) \in S$ and $(a + 3, b + 2) \in S$.

Use structural induction to show that $a + b$ is divisible by 5 when $(a, b) \in S$.

4. **20 pts**

Prove by induction that $x_1 \oplus x_2 \oplus x_3 \ldots \oplus x_n$ is true if and only if an odd number of $x_1, x_2, x_3, \ldots, x_n$ are true (for $n \geq 1$).

5. **20 pts**

State whether $f(x)$ is $O(g(x))$ (or not) and whether $f(x)$ is $\Omega(g(x))$ (or not) and prove your answer.

$f(x) = x^5 + 7x^4 + 77$ and $g(x) = x^6$

6. **10 pts**

Solve the recurrence relation

$$f(n) = 5f(n-1) - 6f(n-2)$$

for $f(0) = 1$ and $f(1) = 0$.

7. 20 pts

Write an algorithm which lists the elements of $A - B$ where A and B are finite sets and A has n elements and B has m elements. You may write your algorithm in English or in pseudocode. What is the time complexity (worst case) in big-O notation in terms of m and n? Justify your answer.

8. **15 pts**
Apply the Master Theorem to give bounds (in big-O notation) for these recurrences:

a. $f(n) = 5f(\frac{n}{2}) + 3n^4$

b. $f(n) = 4f(\frac{n}{2}) + 10n$

c. $f(n) = f(\frac{n}{3}) + 100$

9. **15 pts**

For each function below, say whether it is one-to-one (or not) and whether it is onto (or not). Justify your answer.

a. $f : \mathbb{N} \rightarrow \mathbb{N}$ where $\mathbb{N} = \{0, 1, 2, \ldots\}$
$f(x) = \lfloor \frac{x}{3} \rfloor$

b. $f : \mathbb{R} \rightarrow \mathbb{R}$
$f(x) = \log x$

c. $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$
$f(x) = x^2 + y^2$

10. **20 pts**

Prove that you cannot tile a standard checkerboard (8×8) with the upper left and lower right corners removed with dominoes. (Hint: Colour the dominoes and the checkerboard.)

11. **20 pts**

Assuming that $f_0 = 0$ and $f_1 = 1$ are the beginning of a sequence, f_n, of Fibonacci numbers, prove by induction that

$$f_0 + f_2 + f_4 + \ldots + f_{2n} = f_{2n+1} - 1 \text{ for } n \geq 0.$$

12. **20 pts**

Draw the complete graph with six vertices, K_6.

How many ways are there to colour this graph using two colours?

Prove, using the Pigeonhole Principle, that with any colouring using two colours, every vertex is either connected to three vertices the same colour it is or three vertices of the other colour.