Math/CSE 1019: Discrete Mathematics for Computer Science

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Course page: http://www.cse.yorku.ca/course/1019

Course objectives

We will focus on two major goals:

- Basic tools and techniques in discrete mathematics
  - Mathematical logic and proofs (Ch 1)
  - Set Theory, Functions, Sequences (Ch 2)
  - Simple algorithms (Ch 3)
  - Induction, recursion (Ch 4)
  - Counting techniques (Combinatorics) (Ch 5)
  - Recurrence relations (Ch 7)
  - Introductory graph theory (Ch 9)
- Precise and rigorous mathematical reasoning
  - Writing proofs

Logic

Ch. 1: Introduction to Propositional Logic

- Truth values, truth tables
- Boolean logic: $\vee$ $\wedge$ $\neg$
- Implications: $\rightarrow$ $\leftrightarrow$

Why study propositional logic?

- A formal mathematical “language” for precise reasoning.
- Start with propositions.
- Add other constructs like negation, conjunction, disjunction, implication etc.
- All of these are based on ideas we use daily to reason about things.

Propositions

- Declarative sentence
- Must be either True or False.

Propositions:

- York University is in Toronto
- York University is in downtown Toronto
- All students at York are Computer Sc. majors.

Not propositions:

- Do you like this class?
- There are x students in this class.
- Read Ch 1 before the next class.

Propositions - 2

- Truth value: True or False
- Variables: p,q,r,s,…
- Negation:
- $\neg$p (“not p”)
- Truth tables

p	$\neg$p
T	F
F	T

Negation

- Simple: The Lakers beat the Jazz
- Simple: It is cold
- Harder: The Lakers to win the series in 7 games
- Harder: Toronto gets at least 10 ft of snow each winter

Conjunction, Disjunction

- Conjunction: $p \wedge q$ ["and"]
- Disjunction: $p \vee q$ ["or"]

p	q	$p \wedge q$	$p \vee q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

Examples

- Experience with C++ or Java is required
- Must be proficient in Windows and Linux
- Publish or perish
- Dinner for 2 includes 2 items from column A or 3 items from column B

Exclusive OR (XOR)

- $p \oplus q$ – T if p and q have different truth values, F otherwise
- Colloquially, we often use OR ambiguously – "an entrée comes with soup or salad" implies XOR, but "students can take MATH XXXX if they have taken MATH 2320 or MATH 1019" usually means the normal OR (so a student who has taken both is still eligible for MATH XXXX).

More examples

- If you buy a 2010 model from ???, you get $2000 cashback or 0% financing.
- School closes if there is more than 2 ft of snow or if the wind chill falls below –30F

Conditional

$p \rightarrow q$ ["if p then q"]

- p: *hypothesis/antecedent/premise*,
- q: *conclusion/consequence*

E.g.:

- "If you turn in a homework late, it will not be graded";
- "If you get 100% in this course, you will get an A+".
- TRICKY: Is $p \rightarrow q$ TRUE if p is FALSE? **YES!!**
- Think of "If you get 100% in this course, you will get an A+" as a promise – is the promise violated if someone gets 50% and does not receive an A+?

Examples

- To be a citizen of the US it is sufficient that you were born in the US.
- If today is Friday, 2+3=5
- Definition: $p \rightarrow q$ ["if p then q"] is false when p is true and q is false and true otherwise.

If-then in programming

- if(n<10) A[n] = n;

Conditional - 2

- $p \rightarrow q$ ["if p then q"]
- Truth table:

p	q	$p \rightarrow q$	$\neg p \vee q$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

Note the truth table of $\neg p \vee q$

Logical Equivalence

- $p \rightarrow q$ and $\neg p \vee q$ are logically equivalent
- Truth tables are the simplest way to prove such facts.
- You will learn other ways later.

Contrapositive

- Contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$
- Any conditional and its contrapositive are logically equivalent (have the same truth table) – Check by writing down the truth table.
- E.g. The contrapositive of "If you get 100% in this course, you will get an A+" is "If you do not get an A+ in this course, you did not get 100%".

Converse

- Converse of $p \rightarrow q$ is $q \rightarrow p$
- Not logically equivalent to conditional
- Ex 1: "If you get 100% in this course, you will get an A+" and "If you get an A+ in this course, you scored 100%" are not equivalent.
- Ex 2: If you won the lottery, you are rich.

Other conditionals

Inverse:

- inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$
- How is this related to the converse?

Biconditional:

- "If and only if"
- True if p,q have same truth values, False otherwise. Q: How is this related to XOR?
- Can also be defined as $(p \rightarrow q) \wedge (q \rightarrow p)$

Examples

- For you to win the lottery, it is necessary and sufficient that you have a winning ticket.
- The train runs late on exactly the days when I take it
- You get promoted only if you have connections
- You get promoted if and only if you have connections.

Proof using contrapositive

Prove: If x^2 is even, x is even

- Proof 1: $x^2 = 2a$ for some integer a. Since 2 is prime, 2 must divide x.
- Proof 2: if x is not even, x is odd. Therefore x^2 is odd. This is the contrapositive of the original assertion.

Readings and notes

- Read pages 1-12.
- Think about the notion of truth tables.
- Master the rationale behind the definition of conditionals.
- Practice translating English sentences to propositional logic statements.

Next

Ch. 1.2: Propositional Logic - continued

- Compound propositions, precedence rules
- Tautologies and logical equivalences

Ch. 1.3: Predicate Logic

- Predicates and quantifiers
- Rules of Inference

Compound Propositions

- Example: $p \wedge q \vee r$: Could be interpreted as $(p \wedge q) \vee r$ or $p \wedge (q \vee r)$
- precedence order: $\neg \wedge \vee \rightarrow \leftrightarrow$ (IMP!) (Overruled by brackets)
- We use this order to compute truth values of compound propositions.

Examples

- $(p \to q) \vee (p \to r)$
- $p \to (q \vee r)$
- $p \vee (q \wedge r)$
- $(p \vee q) \wedge (p \vee r)$

Tautology

- A compound proposition that is always TRUE, e.g. $q \vee \neg q$
- Logical equivalence redefined: p,q are logical equivalences if $p \leftrightarrow q$ is a tautology. Symbolically $p \equiv q$.
- Intuition: $p \leftrightarrow q$ is true precisely when p,q have the same truth values.

Exercise

- Is $(\neg p \wedge (p \to q)) \to \neg q$ a tautology?

Manipulating Propositions

- Compound propositions can be simplified by using simple rules.
- Read page 24 - 25.
- Some are obvious, e.g. Identity, Domination, Idempotence, double negation, commutativity, associativity
- Less obvious: Distributive, De Morgan's laws, Absorption

Straightforward rules

- Identity $p \wedge T$, $q \vee F$
- Domination $p \vee T$, $q \wedge F$
- Idempotence $p \vee p$, $p \wedge p$
- Double negation $\neg(\neg p)$
- Commutativity $p \vee q \equiv q \vee p$, etc
- Associativity $p \vee (q \vee r) \equiv (p \vee q) \vee r$, etc

Distributive Laws

$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Intuition (not a proof!) – For the LHS to be true: p must be true and q or r must be true. This is the same as saying p and q must be true or p and r must be true.

$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Intuition (less obvious) – For the LHS to be true: p must be true or both q and r must be true. This is the same as saying p or q must be true and p or r must be true.

Proof: use truth tables.

De Morgan's Laws

$\neg(q \vee r) \equiv \neg q \wedge \neg r$

Intuition – For the LHS to be true: neither q nor r can be true. This is the same as saying q and r must be false.

$\neg(q \wedge r) \equiv \neg q \vee \neg r$

Intuition – For the LHS to be true: $q \wedge r$ must be false. This is the same as saying q or r must be false.

Proof: use truth tables.

Examples

Negate:

- Yoshiko knows Java and Calculus
- James is young and strong
- Rita will move to Oregon or Utah

Using the laws

- Show that $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
- Q: Is $p \to (p \to q)$ a tautology?
- Can use truth tables

Limitations of Propositional Logic

- What can we NOT express using predicates?

  E.g.: How do you make a statement about all even integers?

- A more general language: Predicate logic

Predicate Logic

- A predicate is a proposition that is a function of one or more variables.

  E.g.: P(x): x is an even number. So P(1) is false, P(2) is true,….
- Examples of predicates:
  - Domain ASCII characters - IsAlpha(x) : TRUE iff x is an alphabetical character.
  - Domain floating point numbers - IsInt(x): TRUE iff x is an integer.
  - Domain integers: Prime(x) - TRUE if x is prime, FALSE otherwise.

Quantifiers

- describes the values of a variable that make the predicate true. E.g. $\exists x\, P(x)$
- Domain or universe: range of values of a variable (sometimes implicit)

Two Popular Quantifiers

- Universal: $\forall x\, P(x)$ – "P(x) for all x in the domain"
- Existential: $\exists x\, P(x)$ – "P(x) for some x in the domain" or "there exists x such that P(x) is TRUE".
- Either is meaningless if the domain is not known/specified.
- Examples (domain real numbers)
  - $\forall x\, (x^2 >= 0)$
  - $\exists x\, (x > 1)$
  - $(\forall x > 1)\, (x^2 > x)$ – quantifier with restricted domain

Using Quantifiers

Domain integers:

- Using implications: The cube of all negative integers is negative.

$\forall x\, (x < 0) \rightarrow (x^3 < 0)$

- Expressing sums :

$$\forall n\, \left(\sum_{i=1}^{n} i = n(n+1)/2\right)$$