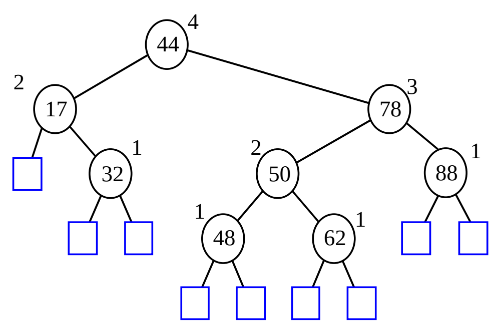
AVL Trees

CSE 2011
Fall 2009

AVL Trees

- AVL trees are balanced.
- An AVL Tree is a ***binary search tree*** such that for every internal node v of T, the *heights of the children of v can differ by at most 1*.



An example of an AVL tree where the heights are shown next to the nodes

Height of an AVL Tree

- **Proposition**: The ***height*** of an AVL tree T storing n keys is O(log n).

Proof:

- Find **n(h):** the *minimum number of internal nodes* of an AVL tree of height h
- We see that n(1) = 1 and n(2) = 2
- For $h \geq 3$, an AVL tree of height h contains the root node, one AVL subtree of height h–1 and the other AVL subtree of height h–2.
- i.e. $n(h) = 1 + n(h-1) + n(h-2)$

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Height of an AVL Tree (2)

- Knowing $n(h-1) > n(h-2)$, we get $n(h) > 2n(h-2)$

  $n(h) > 2n(h-2)$
  $n(h) > 4n(h-4)$
  …
  $n(h) > 2^i n(h-2i)$

- Solving the base case we get: $n(h) \geq 2^{h/2-1}$
- Taking logarithms: $h < 2\log n(h) + 2$
- Thus the height of an AVL tree is O(log n)

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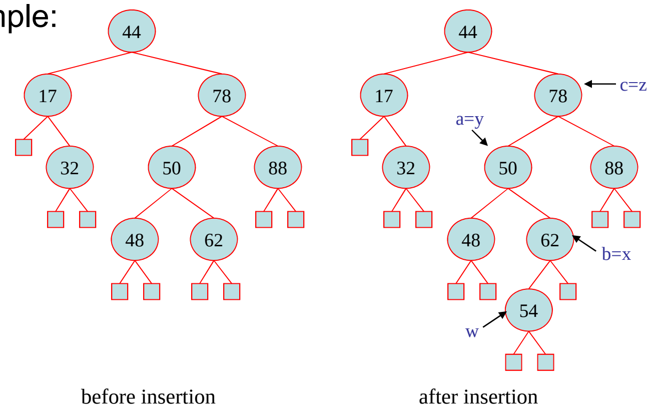
Insertion in an AVL Tree

- Insertion is as in a binary search tree
- Always done by expanding an external node.
- Example:



Insertion (2)

- A binary search tree T is called *balanced* if for every node v, the height of v's children differ by at most 1.
- Inserting a node into an AVL tree involves performing *insertAtExternal*(*w, e*) on T, which changes the heights of some of the nodes in T.
- If an insertion causes T to become unbalanced, we travel up the tree from the newly created node w until we find the first node z that is unbalanced.
- y = child of z with higher height (Note: y = ancestor of w)
- x = child of y with higher height
  (Note: x = ancestor of w or x = w)
- Since z became unbalanced by an insertion in the subtree rooted at its child y, height(y) = height(sibling(y)) + 2

Insertion: rebalancing

- Now to rebalance...
- To rebalance the subtree rooted at z, we must perform a *restructuring*
- We rename x, y, and z to a, b, and c based on the order of the nodes in an in-order traversal (4 possible mappings)
- z is replaced by b, whose children are now a and c whose children, in turn, consist of the four other subtrees formerly children of x, y, and z.

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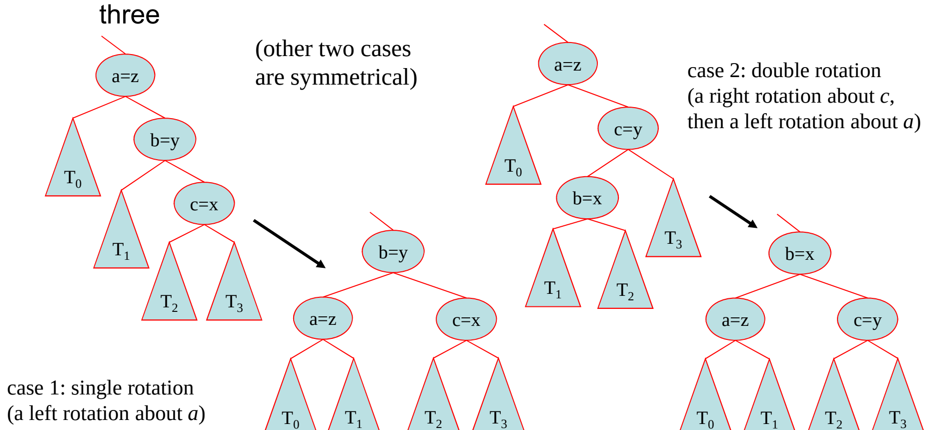
Trinode Restructuring

- let (*a*,*b*,*c*) be an inorder listing of *x*, *y*, *z*
- perform the rotations needed to make *b* the topmost node of the three



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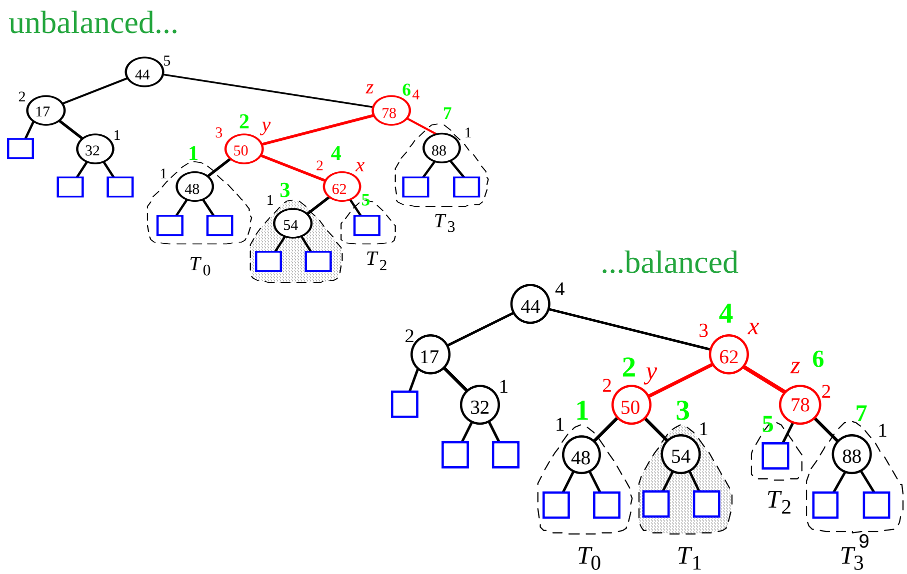
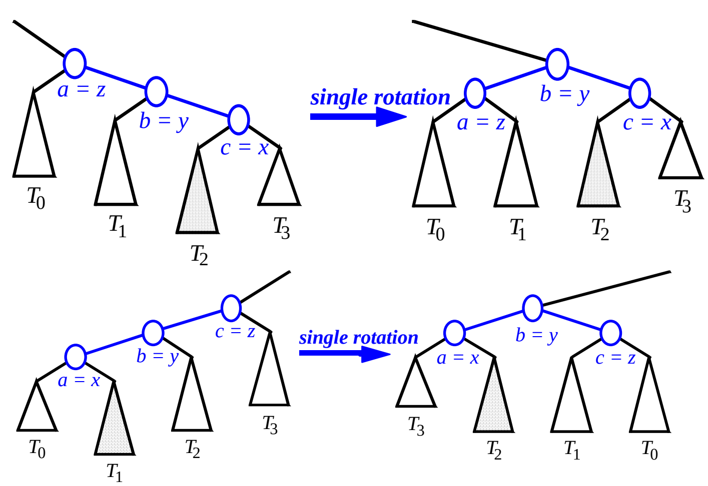
Insertion Example



Restructuring

Single rotations



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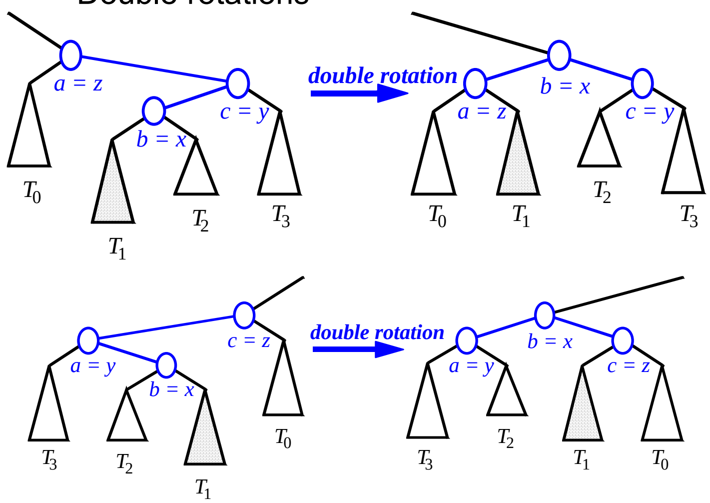
Restructuring (2)

Double rotations



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Restructure Algorithm

Algorithm restructure(x):

Input: A node *x* of a binary search tree T that has both a parent *y* and a grandparent *z*

Output: Tree T restructured by a rotation (either single or double) involving nodes x, y, and z.

1. Let (*a, b, c*) be an inorder listing of the nodes *x, y, and z,* and let (T0, T1, T2, T3) be an inorder listing of the the four subtrees of *x, y, and z,* not rooted at *x, y, or z.*
2. Replace the subtree rooted at z with a new subtree rooted at *b*
3. Let *a* be the left child of *b* and let T0, T1 be the left and right subtrees of *a*, respectively.
4. Let *c* be the right child of *b* and let T2, T3 be the left and right subtrees of *c*, respectively.

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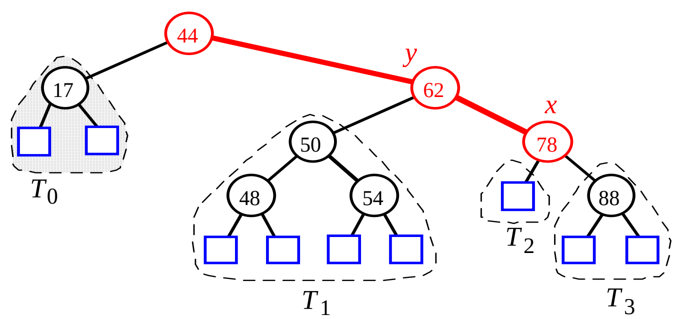
Cut/Link Restructure Algorithm

- Any tree that needs to be balanced can be grouped into 7 parts: x, y, z, and the 4 trees anchored at the children of those nodes (T_0, T_1, T_2, T_3)



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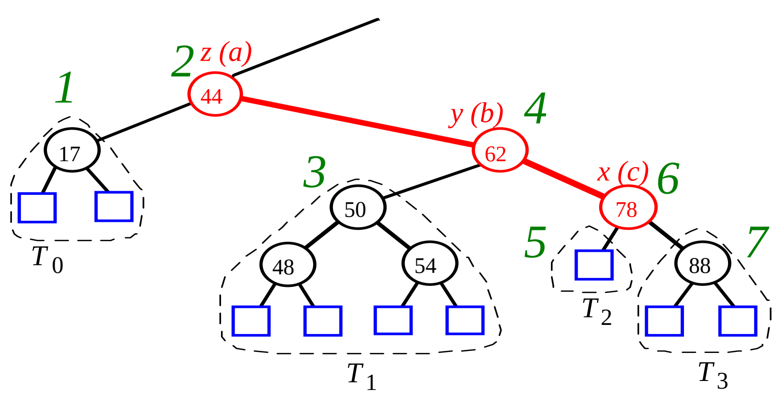
Cut/Link Restructure Algorithm

- Number the 7 parts by doing an inorder traversal
- x,y, and z are now renamed based upon their order within the inorder traversal



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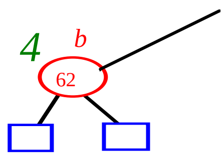
Cut/Link Restructure Algorithm

- Now we can re-link these subtrees to the main tree.
- Link in node 4 (b) where the subtree's root formerly



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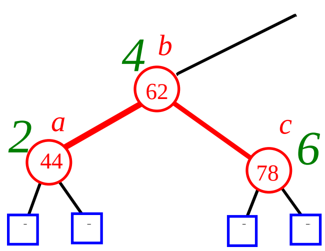
Cut/Link Restructure Algorithm

- Link in nodes 2 (a) and 6 (c) as children of node 4.



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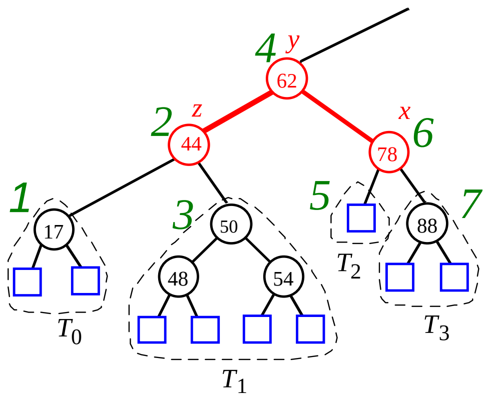
Cut/Link Restructure Algorithm

- Finally, link in subtrees 1 and 3 as the children of node 2, and subtrees 5 and 7 as the children of 6.



Analysis of Cut/Link Restructure Algorithm

- This algorithm for restructuring has the exact same effect as using the four rotation cases discussed earlier.
- Advantages: no case analysis, more elegant
- Disadvantage: can be more code to write
- Same time complexity

Removal

- Performing a removeExternal(w) can cause T to become unbalanced.
- Let *z* be the first unbalanced node encountered while travelling up the tree from w.
- y = child of z with higher height (y ≠ ancestor of w)
- x = child of y with higher height, or either child if two children of y have the same height.
- Perform operation restructure(x) to restore balance at the subtree rooted at z.
- As this restructuring may upset the balance of another node higher in the tree, we must continue checking for balance until the root of T is reached.

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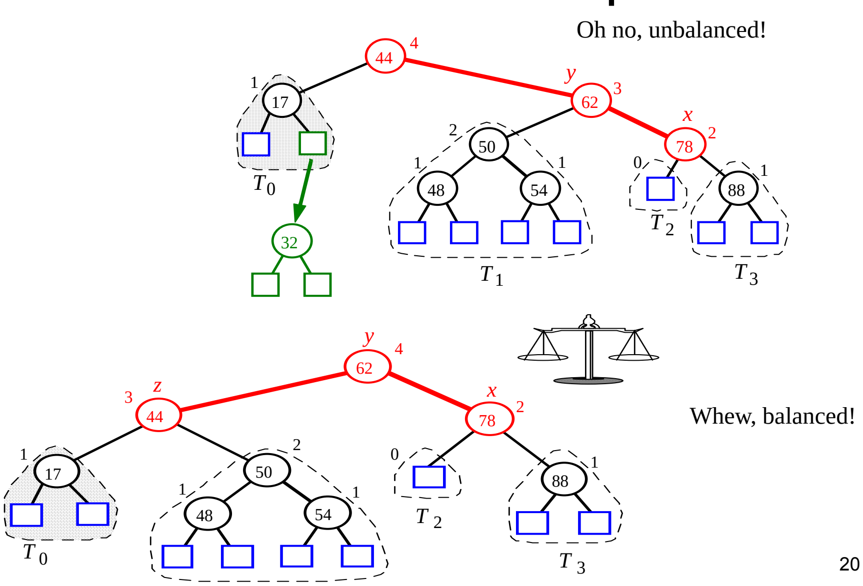
Removal Example



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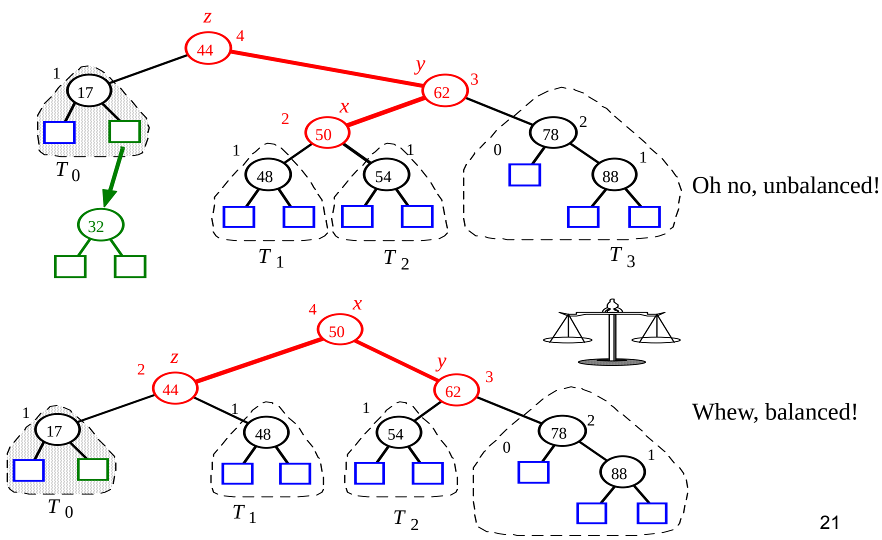
Removal Example (2)



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Next time …

- Heaps (8.3)

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