Homework Assignment #4
Due: April 14, 2009

4. In this question, f and g are functions from the natural numbers to the positive reals such that $f(n) \geq n+1$ and $g(n) \geq n+1$.

(a) Suppose you have a Turing machine M that decides a language L and there is a constant c such that, for all $n \geq c$, M takes at most $f(n)$ steps on each input of size n. Prove that $L \in \mathbf{TIME}(f(n))$.

(b) Suppose there is a constant δ such that $f(n) \geq (1+\delta)n$ and $g(n) \geq (1+\delta)n$ for all n. Show that if $f(n)$ is in $\Theta(g(n))$, then $\mathbf{TIME}(f(n)) = \mathbf{TIME}(g(n))$.

(Recall that $f(n)$ is in $\Theta(g(n))$ iff there exist constants n_0, c_1, c_2 such that $c_1 \geq 0$ and for all $n \geq n_0$, $c_1 g(n) \leq f(n) \leq c_2 g(n)$.)

(c) Let $B(k)$ be the boolean representation of natural number k. Let

$$L = \{x\#B(k) : x \in \{ab, ba\}^*, k \leq |x| \text{ and the } k\text{th character of } x \text{ is } b\}.$$

Prove that $L \in \mathbf{TIME}(1.0001n + 1)$ but $L \notin \mathbf{TIME}(n+1)$.

Hint: For the impossibility result, show that if a TM for L is given an input of the form $x\#B(k)$ with $k \leq |x|$, the TM must read every character of the input tape. Then think about how much information on the work tapes the TM can access after its input head reaches the #.