The Pigeonhole Principle

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The pigeonhole principle

- Assume 13 pigeons fly into 12 pigeonholes to roost.
- A least one of 12 pigeonholes must have at least two pigeons in it.

The pigeonhole principle

The pigeonhole principle:

- $K \in \mathbf{Z}^+$.
- Assume k+1 or more objects are placed into k boxes.
- So, there is at least one box containing two or more of the objects.

Proof by contradiction:

- Suppose none of the k boxes contains more than one object.
- So, the total number of objects would be at most k.
- This is a contradiction, because there are at least k+1 objects.

Example

A function f from set with k+1 or more elements to a set with k elements is not one-to-one.

Proof:

- To use pigeonhole principle, first find boxes and objects.
  - Suppose that for each element y in the codomain of f, we have a box that contains all elements x of the domain of f such that f(x)=y.
  - The number of boxes is k and the number of objects is k+1 or more.
  - By the pigeonhole principle, at least one of these boxes contains two or more elements x of the domain.
  - At least two elements of the domain are assigned to the same element in codomain.
  - So, f cannot be one-to-one.

Example

Show that among any group of 367 people, there must be at least two with the same birthday.

Proof:

- To use pigeonhole principle, first find boxes and objects.
  - Suppose that for each day of a year, we have a box that contains a birthday that occurs on that day.
  - The number of boxes is 366 and the number of objects is 367.
  - By the pigeonhole principle, at least one of these boxes contains two or more birthdays.
  - So, there must be at least two people with the same birthday.

Example

Show that in any group of 27 English words, there must be at least two that begin with the same letter.

Proof:

- To use pigeonhole principle, first find boxes and objects.
  - Suppose that for each letter, we have a box that contains a word that begins with that letter.
  - The number of boxes is 26 and the number of objects is 27.
  - By the pigeonhole principle, at least one of these boxes contains two or more words.
  - So, there must be at least two words that begin with the same letter.

Example

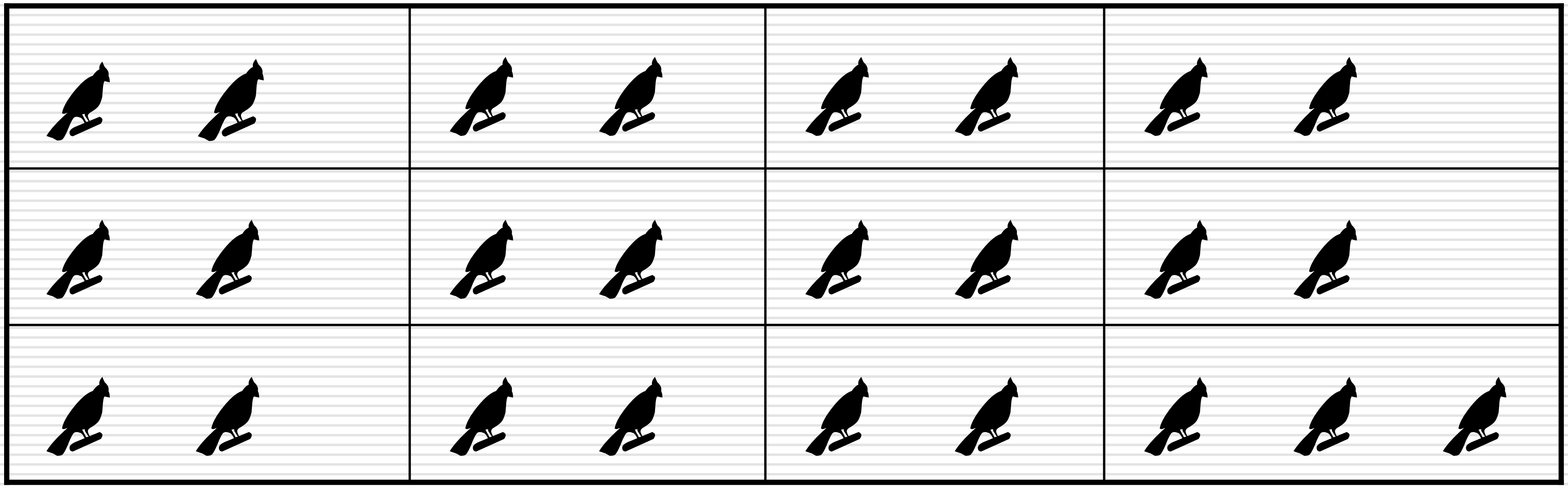
How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points.

Proof:

- To use pigeonhole principle, first find boxes and objects.
  - Suppose that for each score, we have a box that contains a student which got that score in the final exam.
  - The number of boxes is 101, so by the pigeonhole principle, the number of students must be 102 or more.

The generalized pigeonhole principle

- Assume 25 pigeons fly into 12 pigeonholes to roost.
- A least one of 12 pigeonholes must have at least three pigeons in it.



The generalized pigeonhole principle

The generalized pigeonhole principle:

- Assume N objects are placed into k boxes.
- So, there is at least one box containing at least $\lceil N/k \rceil$ objects.

Proof by contradiction:

- Assume none of the boxes contains more than $\lceil N/k \rceil$ -1 objects.
- So, the total number of objects is at most

  $$k(\lceil N/k \rceil -1) < k((N/k+1)-1) = N.$$

- This is a contradiction because there are a total of N objects.

Example

Show among 100 people there are at least 9 who were born in the same month.

Solution:

- To use pigeonhole principle, first find boxes and objects.
  - Suppose that for each month, we have a box that contains persons who was born in that month.
  - The number of boxes is 12 and the number of objects is 100.
  - By the generalized pigeonhole principle, at least one of these boxes contains at least $\lceil 100/12 \rceil = 9$ persons.
  - So, there must be at least 9 persons who were born in the same month.

Example

What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D and F.

Solution:

- To use pigeonhole principle, first find boxes and objects.
  - Suppose that for each grade, we have a box that contains students who got that grade.
  - The number of boxes is 5, by the generalized pigeonhole principle, to have at least 6 (= $\lceil N/5 \rceil$) students at the same box, the total number of the students must be at least N = 5 . 5 + 1 = 26.

Example

Assume there is a standard deck of 52 cards.

a) How many cards must be selected to guarantee that at least three cards of the same suit are chosen?

b) How many cards must be selected to guarantee that at least three hearts are selected

Solution:

Part a:

- Suppose that for each suite, we have a box that contains cards of that suit.
- The number of boxes is 4, by the generalized pigeonhole principle, to have at least 3 (= $\lceil N/4 \rceil$) cards at the same box, the total number of the cards must be at least N = 2 . 4 + 1 = 9.

Example

Assume there is a standard deck of 52 cards.

a) How many cards must be selected to guarantee that at least three cards of the same suit are chosen?

b) How many cards must be selected to guarantee that at least three hearts are selected

Solution:

Part b:

- The worst case, we may selects all the clubs, diamonds, and spades (39 cards) before any hearts.
- So, to guarantee that at least three hearts are selected, 39+3=42 cards should be selected.

Example

Assume that telephone numbers are of the form NXX-NXX-XXXX where N is a digit from 2 to 9 and X can be any digit.

What is the least number of area codes needed to guarantee that 25 million phone numbers can be assigned.

Solution:

- The number of different phone numbers of the form NXX-XXXX is 8 . 10 . 10 . 10 . 10 . 10 . 10 = 8 million.
- Suppose that for each area code, we have a box that contains telephone numbers with that area code.
- The number of boxes is k and the number of telephone numbers is 25,000,000.
- No box contains more than 8,000,000 telephone numbers.

Example

Assume that telephone numbers are of the form NXX-NXX-XXXX where N is a digit from 2 to 9 and X can be any digit.

What is the least number of area codes needed to guarantee that 25 million phone numbers can be assigned.

Solution:

- By the generalized pigeonhole principle, these boxes contains no more than $\lceil 25{,}000{,}000/k \rceil = 8{,}000{,}000$ telephone numbers. ($k = \lceil 25{,}000{,}000/8{,}000{,}000 \rceil$)
- So, k must be at least 4 and 4 area codes are enough.

Recommended exercises

3,5,9,13,21,25,31