Strong Induction and Well-Ordering

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Strong induction

When we cannot easily prove a result using mathematical induction, **strong induction** can often be used to prove the result.

Strong induction

Assume P(n) is a propositional function.

Principle of strong induction:

To prove that P(n) is true for all positive integers n we complete two steps

1. Basis step:

Verify P(1) is true.

2. Inductive step:

Show $[P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k+1)$ is true for all positive integers k.

Strong induction

Basis step: $P(1)$

Inductive step: $\forall k\ ([P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k+1))$

Result: $\forall n\ P(n)$ domain: positive integers

1. $P(1)$
2. $\forall k\ ([P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k+1))$
3. $P(1) \rightarrow P(2)$
4. $P(2)$ by Modus ponens
5. $P(1) \wedge P(2)$
6. $P(1) \wedge P(2) \rightarrow P(3)$
7. $P(3)$ by Modus ponens

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Strong induction VS. mathematical induction

- Strong induction is a more flexible proof technique.
- Mathematical induction and strong induction are equivalent.

Strong induction VS. mathematical induction

- When to use mathematical induction.
  - When it is straightforward to prove P(k+1) from the assumption P(k) is true.

- When to use strong induction.
  - When you can see how to prove P(k+1) from the assumption P(j) is true for all positive integers j not exceeding k.

Example

Show that if n is an integer greater than 1, then n can be written as the product of primes.

Proof by strong induction:

- First define P(n)

  P(n) is n can be written as the product of primes.

- Basis step: (Show P(2) is true.)

  2 can be written as the product of one prime, itself.

  So, P(2) is true.

Example

Show that if n is an integer greater than 1, then n can be written as the product of primes.

Proof by strong induction:

- Inductive step: (Show $\forall k \geq 2\ ([P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k+1))$ is true.)
  - Inductive hypothesis:

    j can be written as the product of primes when $2 \leq j \leq k$.
  - Show $P(k+1)$ is true.
    - Case 1: $(k+1)$ is prime.

      If $k+1$ is prime, $k+1$ can be written as the product of one prime, itself. So, $P(k+1)$ is true.

Example

Show that if n is an integer greater than 1, then n can be written as the product of primes.

Proof by strong induction:

- Case 2: (k+1) is composite.

  $k+1 = a \cdot b$ with $2 \leq a \leq b \leq k$

  By inductive hypothesis, a and b can be written as the product of primes.

  So, k+1 can be written as the product of primes, namely, those primes in the factorization of a and those in the factorization of b.

- We showed that P(k+1) is true.

So, by strong induction $\forall n\, P(n)$ is true.

Example

Game:

- There the two piles of matches.
- Two players take turns removing any positive number of matches they want from one of the two piles.
- The player who removes the last match wins the game.

Show that if two piles contain the same number of matches initially, the second player always guarantees a win.

Example

Proof by strong induction:

- First define P(n)

  P(n) is “player 2 can win when there are initially n matches in each pile”.

- Basis step: (Show P(1) is true.)

  When n=1, player 1 has only one choice, removing one match from one of the piles, leaving a single pile with a single match, which player 2 can remove to win the game.

  So, P(1) is true.

Example

Proof by strong induction:

- Inductive step: (Show $\forall k\,([P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k+1))$ is true.)
  - Inductive hypothesis:

    $P(j)$ is true when $1 \le j \le k$.

    Player 2 can win the game when there are j matches in each pile.
  - Show $P(k+1)$ is true.

    We need to show $P(k+1)$ is true.

    $P(k+1)$ is "Player 2 can win the game when there are $k+1$ matches in each pile".

Example

Proof by strong induction:

- Assume there are k+1 matches in each pile.
- Case 1: Player 1 removes k+1 from one of the piles.
  - Player 2 can win by removing the remaining matches from the other pile.
- Case 2: Player 1 removes r matches from one of the piles. ($1 \le r \le k$).
  - So, k+1-r matches are left in this pile.
  - Player 2 removes r matches from the other pile.
  - Now, there are two piles each with k+1-r matches.
  - Since $1 \le k+1-r \le k$, by inductive hypothesis, Player 2 can win the game.
- We showed that P(k+1) is true.

So, by strong induction $\forall n\, P(n)$ is true.

Strong induction

Sometimes $P(n)$ is true for all integer n with $n \geq b$.

Strong induction:

- **Basis step:**
  - Show $P(b), P(b+1), \ldots, P(b+j)$ are true.
- **Inductive step:**
  - Show $(P(b) \wedge P(b+1) \wedge \ldots \wedge P(k)) \rightarrow P(k+1)$ is true for every positive integer $k \geq b+j$.

Example

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Proof by strong induction:

- First define P(n)

  P(n) is “Postage of n cents can be formed using 4-cent and 5-cent stamps”.

- Basis step: (Show P(12), P(13), P(14) and P(15) are true.)

  P(12) is true, because postage of 12 cents can be formed by three 4-cent stamps.

  P(13) is true, because postage of 13 cents can be formed by three 4-cent stamps and one 5-cent stamp.

  P(14) is true, because postage of 14 cents can be formed by two 5-cent stamps and one 4-cent stamp.

  P(15) is true, because postage of 15 cents can be formed by three 5-cent stamps.

Example

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Proof by strong induction:

- Inductive step: (Show $\forall k \geq 12 \, ([P(12) \wedge P(13) \wedge \ldots \wedge P(k)] \rightarrow P(k+1))$ is true.)
  - Inductive hypothesis:

    $P(j)$ is true when $12 \leq j \leq k$ and $k \geq 15$.

    The postage of j cents can be formed can be formed using just 4-cent and 5-cent stamps.
  - Show $P(k+1)$ is true.

    $P(k+1)$ is "The postage of k+1 cents can be formed using just 4-cent and 5-cent stamps".

Example

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Proof by strong induction:

- Since $12 \le k-3 \le k$, $P(k-3)$ is true by inductive hypothesis.
- So, postage of k-3 cents can be formed using just 4-cent and 5-cent stamps.
- To form postage of k+1 cents, we need only add another 4-cent stamp to the stamps we used to form postage of k-3 cents.

- We showed $P(k+1)$ is true.

So, by strong induction $\forall n\, P(n)$ is true.

Well-ordering

The well-ordering property:

Every non-empty set of nonnegative integers has a least element.

Example

Use the well-ordering property to prove if a is an integer and d is a positive integer, then there are unique integers q and r with $0 \le r < d$ and $a = dq+r$.

Proof by well-ordering:

- We want to show that there are unique q and r.
- $r = a-dq$
- Let S be the set of nonnegative integers of the form a-dq, where q is an integer.
- S is non-empty.
  - If $q<0$, then $a-dq \ge 0$ and $a-dq \in S$.
- By the well-ordering property, S has a least element nonnegative integer $r=a-dq_0$.

Example

Use the well-ordering property to prove if a is an integer and d is a positive integer, then there are unique integers q and r with $0 \le r < d$ and $a = dq+r$.

Proof by well-ordering:

- If $r \ge d$, then $r=d+x$ where $0 \le x < r$.
  - $x+d = a-dq_0$
  - $x = a-d(q_0+1)$
  - So, $x \in S$ and $x < r$ contradicts the well-ordering property (r is a least element of S.)
  - So, $r<d$.
- So, there is integer r (a least element of S) that $r = a-dq$ with $0 \le r < d$.
- Also, there is integer q that $q = (a-r)/d$.
- We showed that there exist integers r and q.
- Show q and r are unique as exercise.

Well-ordering

The well-ordering property, the mathematical induction principle and strong induction are all equivalent

Example

Show that strong induction is a valid method of proof by showing that it follows from the well-ordering property.

Solution:

- Assume we showed $\forall n\ P(n)$ using strong induction.
  - Basis step: $P(1)$ is true.
  - Inductive step: $\forall k\ ([P(1) \wedge P(2) \wedge \ldots \wedge P(k)] \rightarrow P(k+1))$ is true.
- Assume strong induction is not valid (proof by contradiction), so $\exists n\ \neg P(n)$.
- Let S be the set of counterexamples.
  - $S=\{n \mid \neg P(n)\}$
- So, $S \neq \emptyset$.
- By well-ordering property, S has a least element x.
- Since by basis step $P(1)$ is true, $1 \notin S$ and $x \neq 1$.

Example

Show that strong induction is a valid method of proof by showing that it follows from the well-ordering property.

Solution:

- So, $x>1$ and $P(x)$ is false, since $x \in S$.
- Also, $\forall j < x$, $P(j)$ is true.
- By inductive step, $(P(1) \wedge P(2) \wedge \ldots \wedge P(x-1)) \rightarrow P(x)$.
- $P(1) \wedge P(2) \wedge \ldots \wedge P(x-1)$ is true, so by Modus ponens $P(x)$ is true (which contradicts the fact that $x \in S$).
- So, $S = \varnothing$.

Recommended exercises

3,7,11,14,26,29,30,31,32,35,42