Nested Quantifiers

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Nested quantifiers

Two quantifiers are nested if one is within the scope of the other.

Example:

$$\forall x \underbrace{\exists y \overbrace{(x + y = 0)}^{P(x,y)}}_{Q(x)}$$

$\forall x\ Q(x)$

$Q(x)$ is $\exists y\ P(x,y)$

$P(x,y)$ is $(x + y = 0)$

Nested quantifiers (example)

Translate the following statement into English.

$\forall x \, \forall y \, (x + y = y + x)$

Domain: real numbers

Solution:

For all real numbers x and y, $x + y = y + x$.

Nested quantifiers (example)

Translate the following statement into English.

$\forall x \; \exists y \; (x = -y)$

Domain: real numbers

Solution:

For every real number x, there is a real number y such that $x = -y$.

Nested quantifiers (example)

Translate the following statement into English.

$$\forall x \, \forall y \, ((x > 0) \land (y < 0) \rightarrow (xy < 0))$$

Domain: real numbers

Solution:

For every real numbers x and y, if x is positive and y is negative then xy is negative.

The product of a positive real number and a negative real number is always a negative real number.

The order of quantifiers (example)

Assume $P(x,y)$ is $(xy = yx)$.

Translate the following statement into English.

$\forall x \, \forall y \, P(x,y)$ domain: real numbers

Solution:

For all real numbers x, for all real numbers y, $xy = yx$.

For every pair of real numbers x, y, $xy = yx$.

The order of quantifiers

The order of nested **universal** quantifiers in a statement *without* other quantifiers can be changed without changing the meaning of the quantified statement.

The order of quantifiers (example)

Assume $P(x,y)$ is $(xy = 6)$.

Translate the following statement into English.

$\exists x \exists y P(x,y)$ domain: integers

Solution:

There is an integer x for which there is an integer y that $xy = 6$.

There is a pair of integers x, y for which $xy = 6$.

The order of quantifiers

The order of nested **existential** quantifiers in a statement *without* other quantifiers can be changed without changing the meaning of the quantified statement.

The order of quantifiers (example)

Assume $P(x,y)$ is $(x + y = 10)$.

$\forall x \exists y P(x,y)$ domain: real numbers

For all real numbers x there is a real number y such that $x + y = 10$.

True $(y = 10 - x)$

$\exists y \forall x P(x,y)$ domain: real numbers

There is a real number y such that for all real numbers x, $x + y = 10$.

False

So, $\forall x \exists y P(x,y)$ and $\exists y \forall x P(x,y)$ are not logically equivalent.

The order of quantifiers

Assume P(x,y,z) is $(x + y = z)$.

$\forall x\ \forall y\ \exists z\ P(x,y,z)$ domain: real numbers

For all real numbers x and y there is a real number z such that $x + y = z$.

True

$\exists z\ \forall x\ \forall y\ P(x,y,z)$ domain: real numbers

There is a real number z such that for all real numbers x and y $x + y = z$.

False

So, $\forall x\ \forall y\ \exists z\ P(x,y,z)$ and $\exists z\ \forall x\ \forall y\ P(x,y,z)$ are not logically equivalent.

The order of quantifiers

The **order** of nested existential and universal quantifiers in a statement is important.

Quantification of two variable

- **$\forall x \, \forall y \, P(x,y)$**
  - When true?

    P(x,y) is true for every pair x,y.
  - When false?

    There is a pair x, y for which P(x,y) is false.

- **$\forall x \, \exists y \, P(x,y)$**
  - When true?

    For every x there is a y for which P(x,y) is true.
  - When false?

    There is an x such that P(x,y) is false for every y.

Quantification of two variable

- **$\exists x \, \forall y \, P(x,y)$**
  - When true?

    There is an x for which P(x,y) is true for every y.
  - When false?

    For every x there is a y for which P(x,y) is false.

- **$\exists x \, \exists y \, P(x,y)$**
  - When true?

    There is a pair x, y for which P(x,y) is true.
  - When false?

    P(x,y) is false for every pair x, y.

Nested quantifiers (example)

Translate the following statement into a logical expression.

"The sum of two positive integers is always positive."

Solution:

- Rewrite it in English that **quantifiers** and a **domain** are shown

  "For **every** pair of **integers**, if both integers are positive, then the sum of them is positive."

Nested quantifiers (example)

Translate the following statement into a logical expression.
“The sum of two positive integers is always positive.”

Solution:

- Introduce variables

  “For every pair of integers, if both integers are positive, then the sum of them is positive.”

  “For all integers x, y, if x and y are positive, then x+y is positive.”

Nested quantifiers (example)

Translate the following statement into a logical expression.

"The sum of two positive integers is always positive."

Solution:

- Translate it to a logical expression

  "For all integers x, y, if x and y are positive, then x+y is positive."

$\forall x\ \forall y\ ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$ domain: integers

$\forall x\ \forall y\ (x + y > 0)$ domain: positive integers

Nested quantifiers (example)

Translate the following statement into a logical expression.

“Every real number except zero has a multiplicative inverse.”

A multiplicative inverse of a real number x is a real number y such that xy = 1.

Solution:

- Rewrite it in English that **quantifiers** and a **domain** are shown

  “For **every real number** except zero, **there is** a multiplicative inverse.”

Nested quantifiers (example)

Translate the following statement into a logical expression.

“Every real number except zero has a multiplicative inverse.”

A multiplicative inverse of a real number x is a real number y such that xy = 1.

Solution:

- Introduce variables

  “For every real number except zero, there is a multiplicative inverse.”

  “For every real number x, if $x \neq 0$, then there is a real number y such that xy = 1.”

Nested quantifiers (example)

Translate the following statement into a logical expression.

"Every real number except zero has a multiplicative inverse."

A multiplicative inverse of a real number x is a real number y such that xy = 1.

Solution:

- Translate it to a logical expression

  "For every real number x, if $x \neq 0$, then there is a real number y such that $xy = 1$."

  $$\forall x\, ((x \neq 0) \rightarrow \exists y\, (xy = 1))$$ **domain: real numbers**

Nested quantifiers (example)

Translate the following statement into English.

$\forall x\, (C(x) \vee \exists y\, (C(y) \wedge F(x,y)))$

C(x): x has a computer.

F(x,y): x and y are friends.

Domain of x and y: all students

Solution:

"For every student x, x has a computer or there is a student y such that y has a computer and x and y are friends."

"Every student has a computer or has a friend that has a computer."

Nested quantifiers (example)

Translate the following statement into English.

$\exists x \, \forall y \, \forall z \, ((F(x,y) \wedge F(x,z) \wedge (y \neq z)) \rightarrow \neg F(y,z))$

F(x,y): x and y are friends.

Domain of x, y and z: all students

Solution:

"There is a student x such that for all students y and all students z, if x and y are friends, x and z are friend and z and y are not the same student, then y and z are not friend."

"There is a student none of whose friends are also friends with each other."

Nested quantifiers (example)

Translate the following statement into logical expression.

"If a person is a student and is computer science major, then this person takes a course in mathematics. "

Solution:

- Determine individual propositional functions
  - S(x): x is a student.
  - C(x): x is a computer science major.
  - T(x,y): x takes a course y.
- Translate the sentence into logical expression

  $\forall x\ ((S(x) \wedge C(x)) \rightarrow \exists y\ T(x,y))$

  Domain of x: all people

  Domain of y: all courses in mathematics

Nested quantifiers (example)

Translate the following statement into logical expression.

“Everyone has exactly one best friend. ”

Solution:

- Determine individual propositional function
  - B(x,y): y is the best friend of x.
- Express the English statement using **variable** and **individual propositional function**
  - For all x, there is y who is the best friend of x and for every person z, if person z is not person y, then z is not the best friend of x.
- Translate the sentence into logical expression

  $\forall x\, \exists y\, (B(x,y) \wedge \forall z\, ((z \neq y) \rightarrow \neg B(x,z))$

  Domain of x, y and z: all people

Nested quantifiers (example)

Translate the following statement into logical expression.
“Everyone has exactly one best friend. ”

Solution:

- Determine individual propositional function
  - B(x,y): y is the best friend of x.
- Express the English statement using **variable** and **individual propositional function**
  - For all x, there is y who is the best friend of x and for every person z, if person z is not person y, then z is not the best friend of x.
- Translate the sentence into logical expression

  $\forall x\ \exists y\ \forall z\ (\ (B(x,y) \land B(x,z)) \rightarrow (y = z)\ )$

  Domain of x, y and z: all people

Nested quantifiers (example)

Translate the following statement into logical expression.

"There is a person who has taken a flight on every airline in the world. "

Solution:

- Determine individual propositional function
  - $F(x,f)$: x has taken flight f.
  - $A(f,a)$: flight f is on airline a.
- Translate the sentence into logical expression

  $\exists x \, \forall a \, \exists f \, (F(x,f) \wedge A(f,a))$

  Domain of x: all people
  Domain of f: all flights
  Domain of a: all airlines

Nested quantifiers (example)

Translate the following statement into logical expression.

“There is a person who has taken a flight on every airline in the world. ”

Solution:

- Determine individual propositional function
  - R(x,f,a): x has taken flight f on airline a.
- Translate the sentence into logical expression

  $\exists x\ \forall a\ \exists f\ R(x,f,a)$

  Domain of x: all people

  Domain of f: all flights

  Domain of a: all airlines

Negating quantified expressions (review)

$\neg \forall x\, P(x)$	$\exists x\, \neg P(x)$
$\neg \exists x\, P(x)$	$\forall x\, \neg P(x)$

Negating nested quantifiers

- Rules for negating statements involving a single quantifiers can be applied for negating statements involving nested quantifiers.

Negating nested quantifiers (example)

What is the negation of the following statement?

$\forall x\ \exists y\ (x = -y)$

Solution:

$\neg\ \forall x\ P(x)$ $\qquad$ $P(x) = \exists y\ (x = -y)$

$\exists x\ \neg P(x)$

$\exists x\ (\neg\ \exists y\ (x = -y))$

$\exists x\ (\forall y\ \neg(x = -y))$

$\exists x\ \forall y\ (x \neq -y)$

Negating nested quantifiers (example)

Translate the following statement in logical expression?

"There is not a person who has taken a flight on every airline."

Solution:

- Translate the positive sentence into logical expression
  - **$\exists x\ \forall a\ \exists f\ (F(x,f) \wedge A(f,a))$** by previous example

    F(x,f): x has taken flight f. A(f,a): flight f is on airline a.
- Find the negation of the logical expression

  $\neg\ \exists x\ \forall a\ \exists f\ (F(x,f) \wedge A(f,a))$

  $\forall x\ \neg\ \forall a\ \exists f\ (F(x,f) \wedge A(f,a))$

  $\forall x\ \exists a\ \neg\ \exists f\ (F(x,f) \wedge A(f,a))$

  $\forall x\ \exists a\ \forall f\ \neg\ (F(x,f) \wedge A(f,a))$

  $\forall x\ \exists a\ \forall f\ (\neg F(x,f) \vee \neg A(f,a))$

Recommended exercises

1,3,10,13,23,25,27,33,39