Review Questions

1. Prove that $\neg(\neg p \Leftrightarrow (r \vee p))$ is logically equivalent to $r \Rightarrow p$.

2. Use a proof by contradiction to show that if $a + b < 2n$ then $a < n$ or $b < n$.

3. Is the following statement true or false?
$\forall x$ in $\mathbb{R}, \exists y$ in $\mathbb{R}$ such that $y \geq 0 \wedge (y = x \vee (x < 0 \wedge y = -x))$.
Explain why your answer is correct.

4. Is the following statement true or false?
For all sets A, B and C, $A - (B \cup C) \subseteq (A - B) \cap (A - C)$.
Prove your answer is correct.

5. Let $f : B \to C$ and $g : A \to B$. Prove that if $f \circ g$ is onto then f is onto.

6. Prove that $x^3 - 10x^2$ is *not* $O(x^2)$.

7. Prove that for every positive integer n, $\sum_{k=1}^{n} k2^k = (n-1)2^{n+1} + 2$.

8. Yark University has 40000 students. Each student takes 5 classes each term. The University offers 1000 classes each term. The largest classroom at Yark holds 180 students. Is this a problem? Explain why.

9. Consider the domain of all people.
Let $P(x, y)$ represent the statement "x is the parent of y".

(a) Translate the following formulas into clear and precise English.

$\exists x \forall y P(x, y)$

$\forall y \exists x P(x, y)$

(b) Express the statement "Somebody has no grandchildren" using only the predicate P.

10. Prove that for all integers a, b and c, the product of some pair of the three integers is non-negative. (In other words, show that $ab \geq 0$ or $ac \geq 0$ or $bc \geq 0$.)

11. Let $A = \{0, 1, 2\}$ and $B = \{1, 3\}$. List the elements of each of the following sets.

(a) $A - B =$

(b) $A \times B =$

12. Let $f : A \to B$. Let S and T be subsets of A.

(a) Prove that $f(S \cap T) \subseteq f(S) \cap f(T)$.

(b) Give an example of a function f and sets S and T such that $f(S \cap T) \neq f(S) \cap f(T)$. Briefly explain why your answer is correct.

13. Give a good lower bound on $\sum_{i=1}^{n} i^{1.5}(\log_2 i)^2$ using Ω notation. Prove your answer is correct.

14. Use the integral method to get a good upper bound on $\sum_{i=0}^{n} ie^{2i^2}$.

15. Is $\Omega(n) \subseteq \Omega(n + \sqrt{n})$? Prove your answer is correct.

16. Is $3^{\log_e n} \in O(2^{\log_e n})$? Prove your answer is correct.

17. Beside each function $f(n)$ in the left column, write down the number of the **first** function $g(n)$ in the right column such that $f(n) \in O(g(n))$.

(a) $n^{1.9}(\log_2 n)^2$ ______

(b) $\sum_{i=1}^{n} \sqrt{i} \log_2 i$ ______

(c) $(n\sqrt{n} + 7n + 2)^2$ ______

(d) $10 \cdot 2^{\log_4 n}$ ______

(e) $\sum_{i=1}^{n} \frac{n \log n}{i}$ ______

1. 1
2. $\log_2 n$
3. $\sqrt{n}$
4. $\sqrt{n} \log_2 n$
5. n
6. $n \log_2 n$
7. n^2
8. n^3
9. 2^n
10. n^n

18. Prove $n^2 \leq 2^n$ for all natural numbers $n \geq 4$.

19. Let $f\colon A \to B$ be a function. For any set $C \subseteq B$, define $f^{-1}(C)$ to be the set $\{a \in A : f(a) \in C\}$. Prove that for every $f\colon A \to B$ and subsets S and T of B we have $f^{-1}(S) \cap f^{-1}(T) = f^{-1}(S \cap T)$.

20. Show that if 51 distinct numbers are chosen among $\{1, 2, 3, \ldots, 100\}$ then there must be two numbers among them whose sum is 101.