AVL Trees

Dynamic Tree Balancing

Problems with BST

- With random insertions and deletions BST has
  Θ (log N) times for search, insert and remove
- But worst case behaviour is Θ (N)
- Problem is that BST's can become unbalanced
- We need a **rebalance operation** on a BST to restore the balance property and regain Θ (log N)
- Rebalancing should be cheap enough that we could do it **dynamically** on every insert and remove
  - » **Preference is to have Θ (1) rebalance time**

AVL Balance Definition

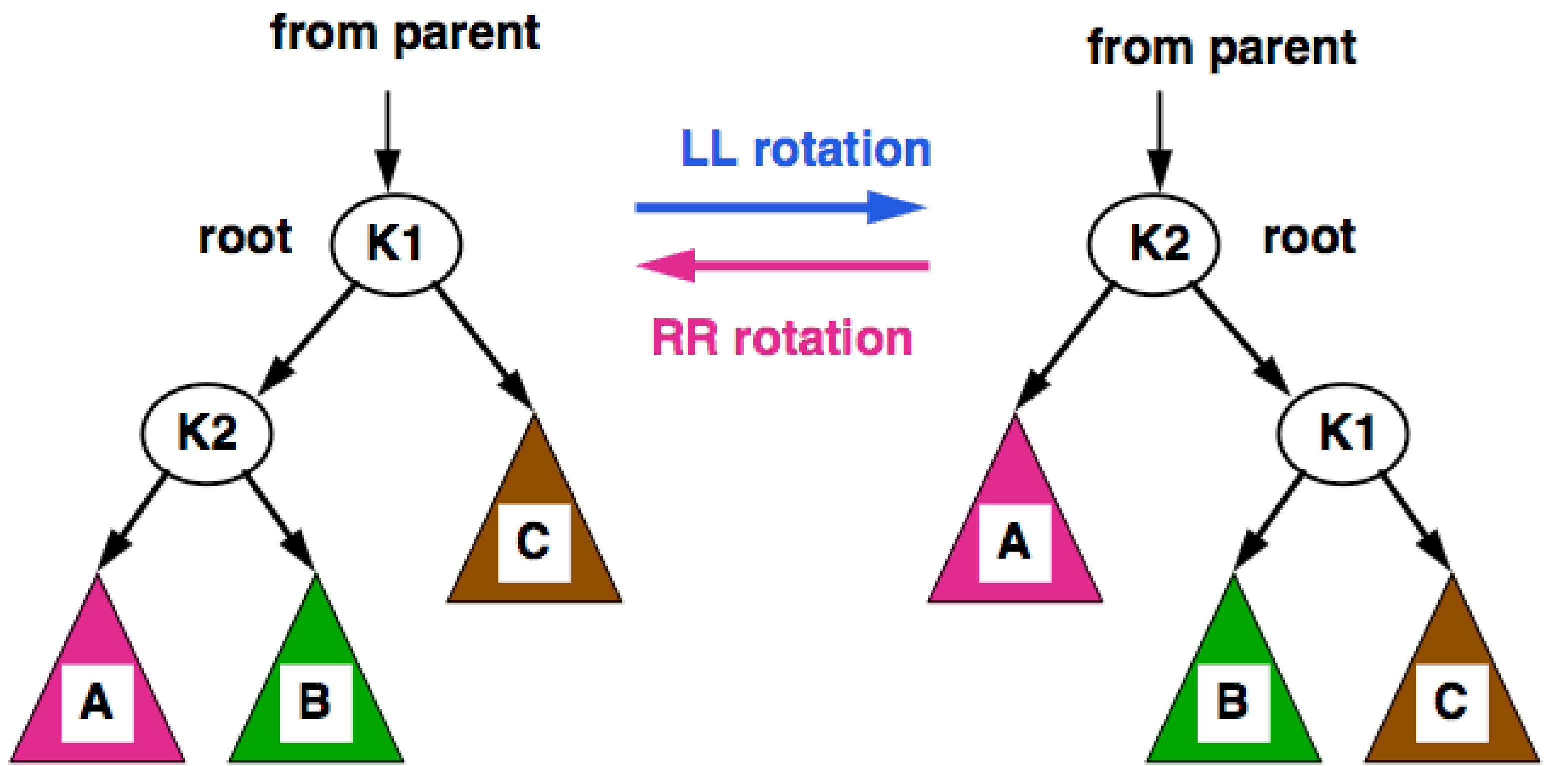
- A good balance conditions ensures the height of a tree with N nodes is Θ (log N)
  - » **That gives Θ (log N) performance**
- The following balance definition is used
  - » **The empty tree is balanced**
  - » **For every node in a non-empty tree**

| height (left_sub_tree) – height (right_sub_tree) | ≤ 1

Rebalancing

- Restructure the tree by moving the nodes around while preserving the order property
- The operation is called a **rotation**
  - » **Make use of the property that a node has one parent and two direct descendents**

Single Rotations



Keys in A < K2
K2 < Keys in B < K1
K1 < Keys in C
Relationship to parent does not change

Single LL Rotation Pseudocode

```
// Return pointer to root after rotation

rotate_LL ( oldRoot : Node ) : Node is
    Result ← oldRoot . left
    oldRoot . left ← Result . right
    Result . right ← oldRoot
    adjustHeight(old_root)
    adjustHeight(old_root.left)
    adjustHeight(Result)
end

// Example use of rotate_LL

parent . left ← rotate_LL ( parent . left)
parent . right ← rotate_LL ( parent . right)
```

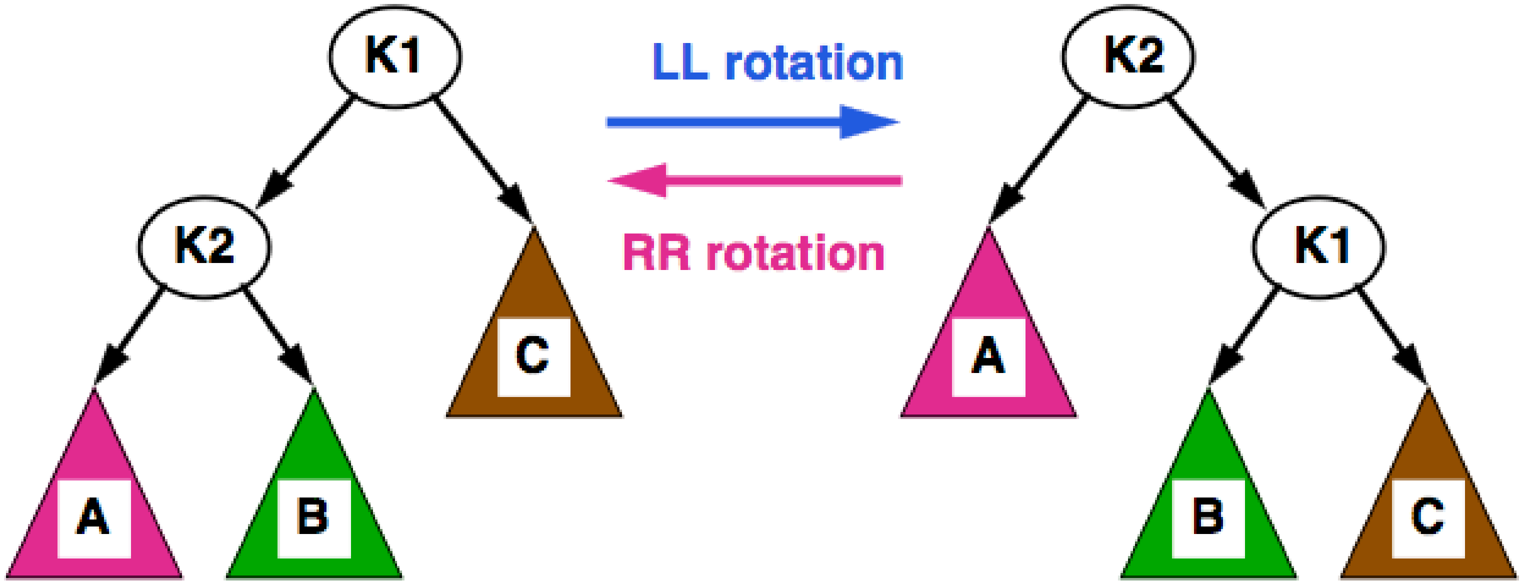
Exercise
write rotate_RR

AdjustHeight Pseudocode

```
// Assume that every node contains a height attribute

adjustHeight ( root : Node ) is
  if root ≠ null then
     root . height ← 1 + max ( height ( root . left )
                             , height ( root . right ) )
  end
end
```

Single Rotations & Height



h (K1) = 1 + max (h(K2) , h(C))
h (K2) = 1 + max (h(A) , h(B))

h (K2) = 1 + max (h(K1) , h(A))
h (K1) = 1 + max (h(B) , h(C))

If $h(A) > h(B)$ & $h(B) \geq h(C)$ then rotate_LL reduces the height of the root

If $h(C) > h(B)$ & $h(B) \geq h(A)$ then rotate_RR reduces the height of the root

Single Rotations & Height – 2

h (K1) = 1 + max (h(K2) , h(C))
h (K2) = 1 + max (h(A) , h(B))

h (K2) = 1 + max (h(K1) , h(A))
h (K1) = 1 + max (h(B) , h(C))

if h(A) > h(B) ∧ h(B) ≥ h(C)
then rotate_LL reduces the height of the root

Proof – before rotation

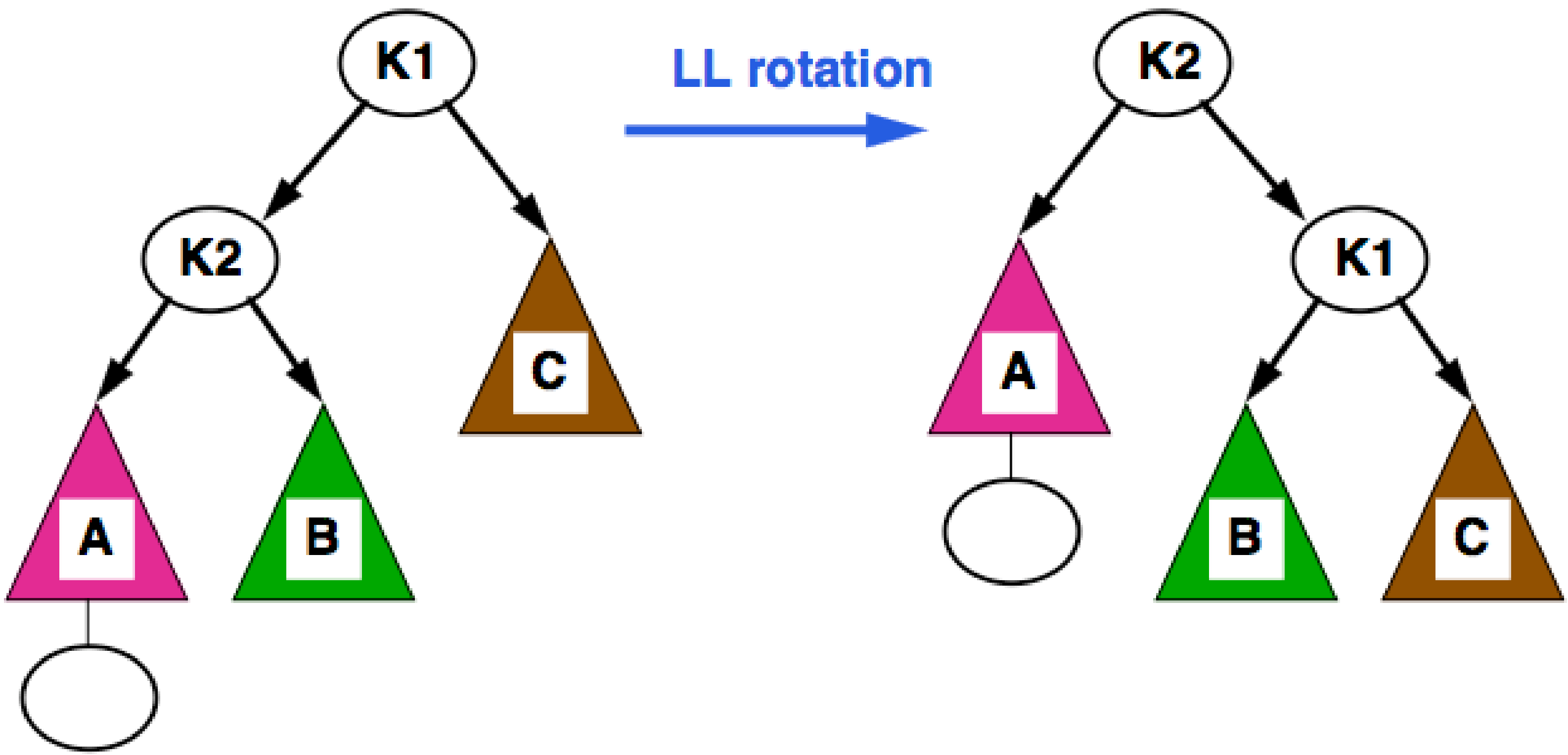
h(K2) = 1 + h(A)
-- h(A) > h(B)
h(K1) = 1 + h(K2)
-- h(K2) > h(B) ≥ h(C)
h(K1) = 2 + h(A)

– after rotation

h(K1) = 1 + h(B)
-- h(B) ≥ h(C)
h(K2) = 1 + h(A)
-- h(A) ≥ 1 + h(B) > h(B)

Before rotation h(root) = 2 + h(A)
After rotation h(root) = 1 + h(A)
Height of root has been reduced

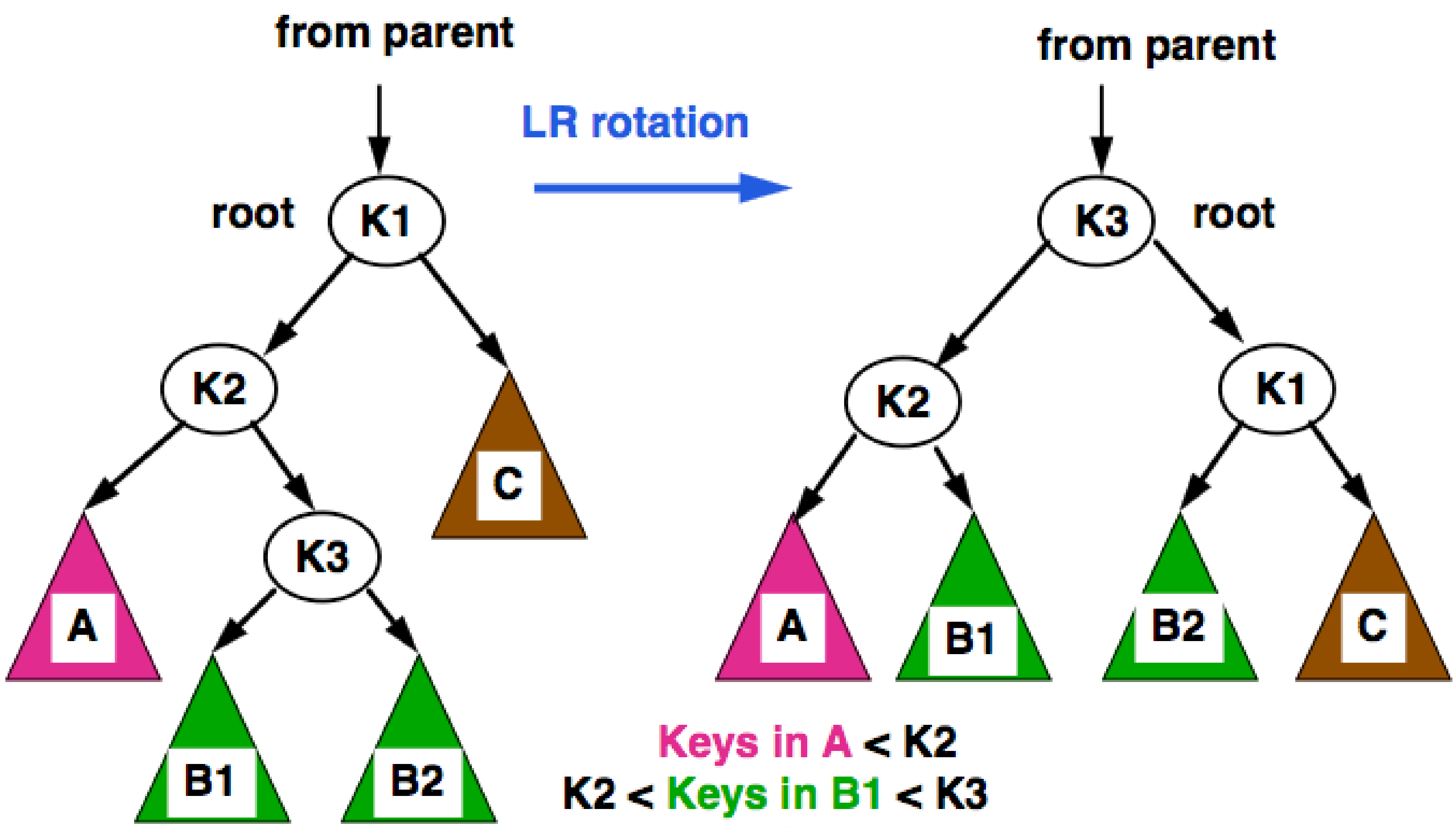
Single Rotations & Height – 3



Proof (?) by diagram

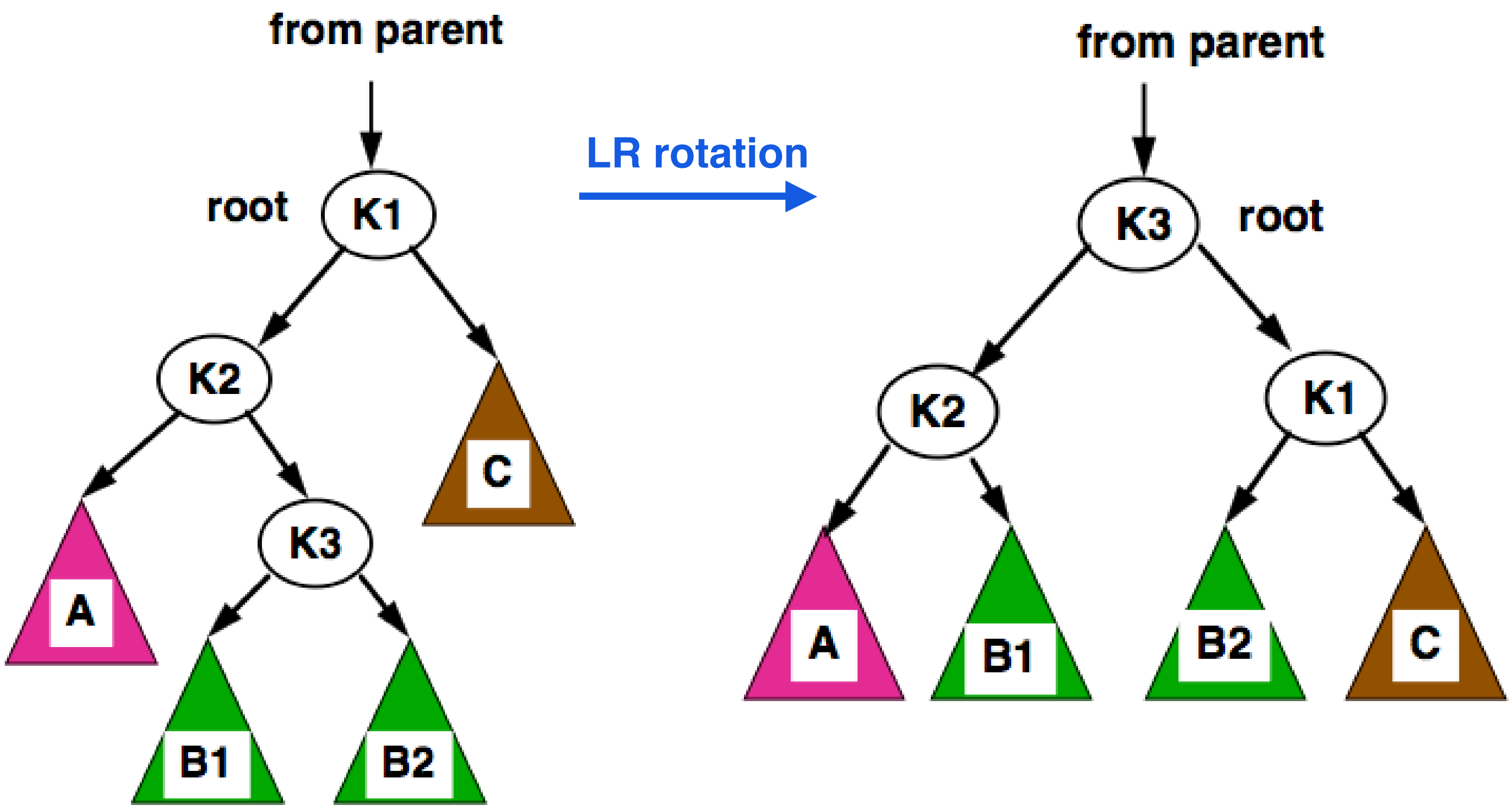
if $h(A) > h(B) \wedge h(B) \geq h(C)$
then rotate_LL reduces the height of the root

Double Rotation – LR



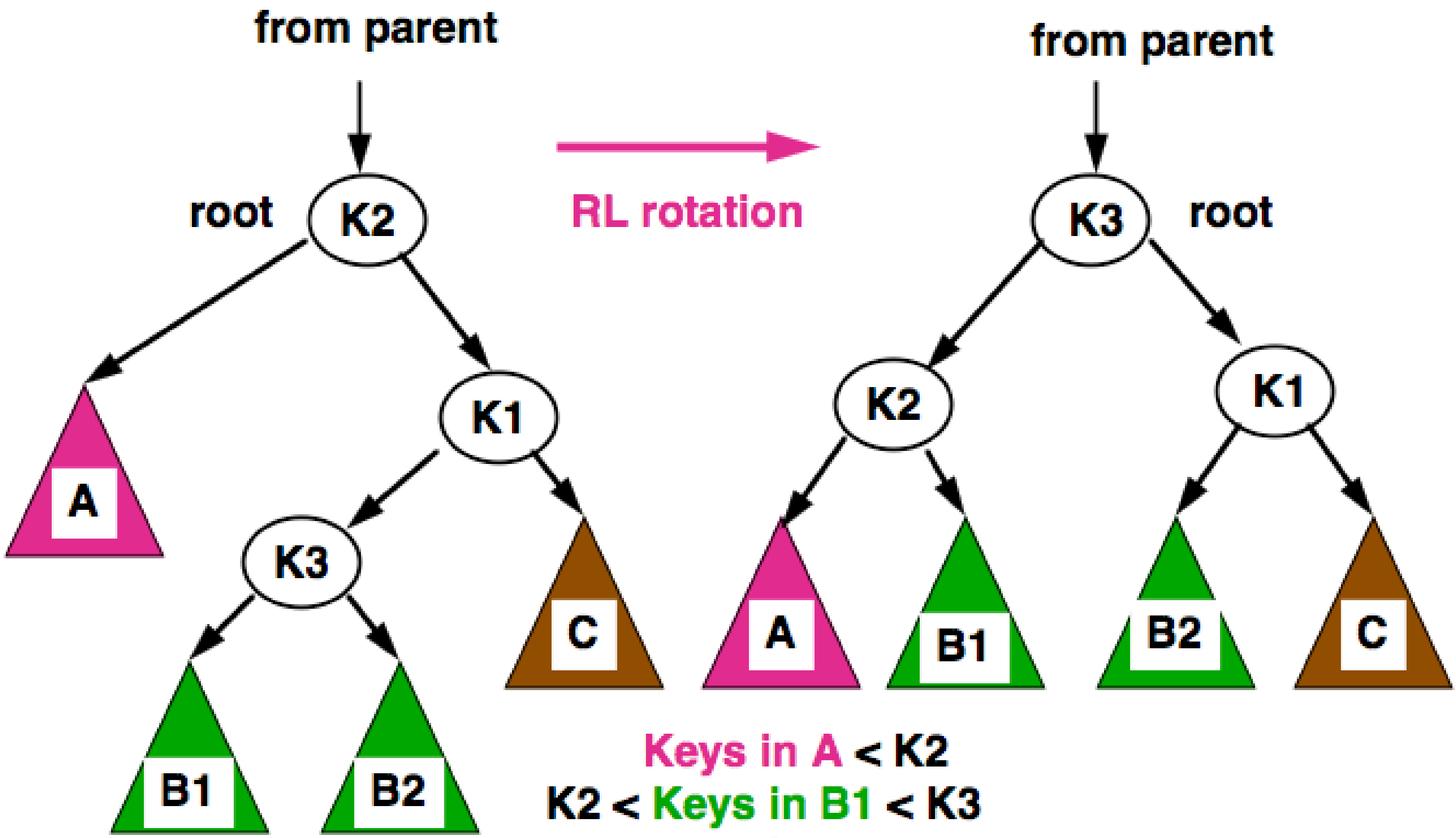
Keys in A < K2
K2 < Keys in B1 < K3
K3 < Keys in B2 < K1
K1 < Keys in C
Relationship to parent does not change

Double Rotation – LR – Height



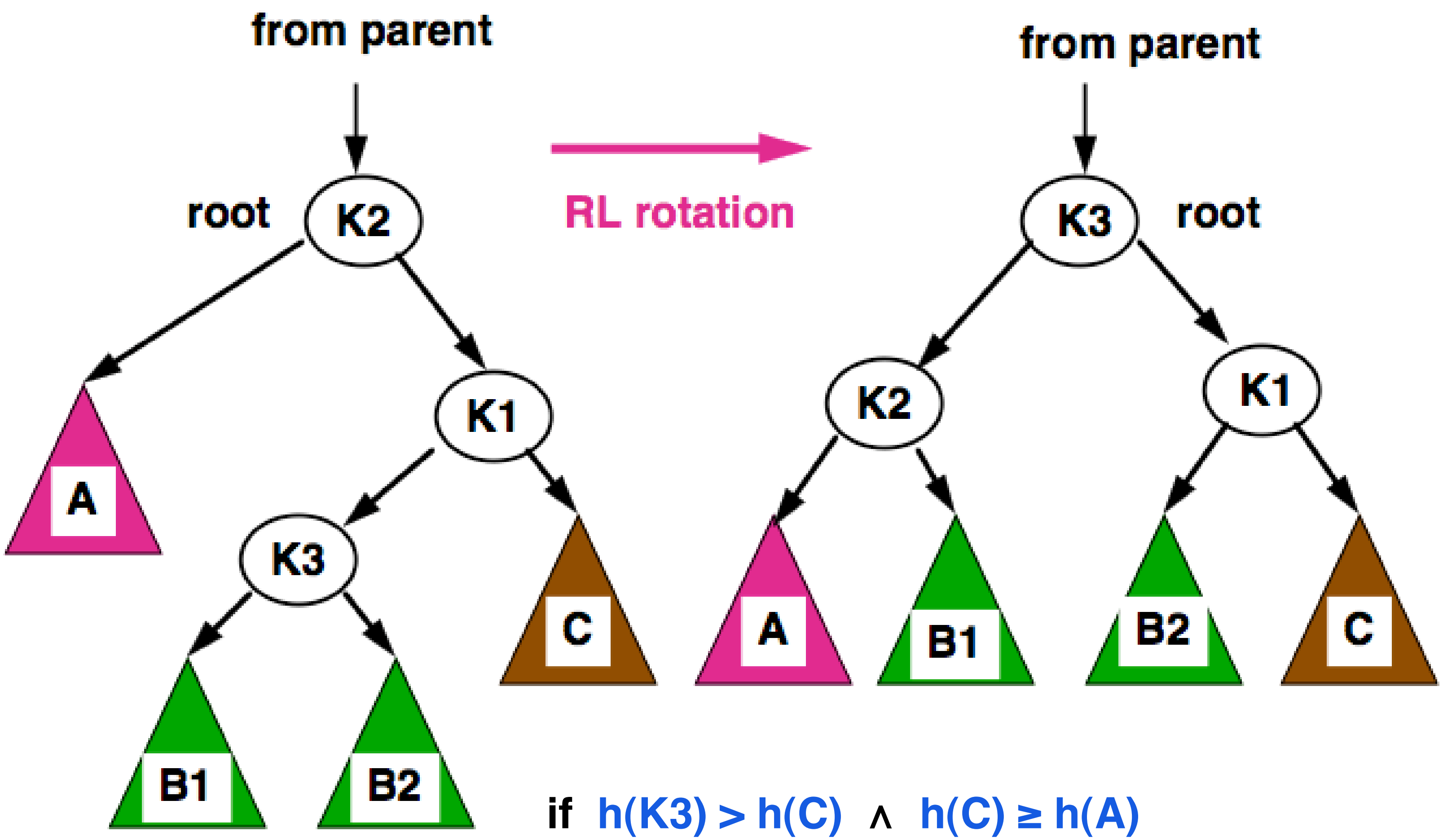
If $h(K3) > h(A) \wedge h(A) \geq h(C)$
then rotate_LR reduces the height of root

Double Rotation – RL



Keys in A < K2
K2 < Keys in B1 < K3
K3 < Keys in B2 < K1
K1 < Keys in C
Relationship to parent does not change

Double Rotation – RL – Height



if $h(K3) > h(C) \wedge h(C) \geq h(A)$
then rotate_RL reduces the height of root

Single RL Rotation Pseudocode

```
// Return pointer to root after rotation

 rotate_RL ( oldRoot : Node ) : Node is
    rightChild ← oldRoot . right  ;  Result ← rightChild . left
    oldRoot . right ← Result . left  ;  rightChild . left ← Result . right
    Result . left ← oldRoot  ;  Result . right ← rightChild

    adjustHeight ( oldRoot )
    adjustHeight ( rightChild )
    adjustHeight ( Result )
end

 // Example use of rotate_RL

 parent . left ← rotate_RL ( parent . left)
 parent . right ← rotate_RL ( parent . right)
```

Exercise
write rotate_LR

Insert into AVL Pseudocode

```
// Insert will do rotations, which changes the root of
// sub-trees.  As a consequence, the recursive insert must
// return the root of the resulting sub-tree.

insert ( key : KeyType , data : ObjectType ) is
   newNode ← new Node ( key , data )
   root ← insertRec ( root , newNode )
   root ← rebalance ( root )      // Insertion may change
   adjustHeight (root)            // height, which may
                                  // cause imbalance
end
```

Only one rebalance will occur but we do not know where

InsertRec Pseudocode

```
// Insert may do rotations, which changes the root of
// sub-trees.  As a consequence, the recursive insert must
// return the root of the resulting sub-tree.

// Invariant –  The tree rooted at root is balanced

 insertRec ( root : Node , newNode : Node ) : Node is
   if root = Void then Result ← newNode
   else if root . key > newNode . key
         then  root . left ← insertRec ( root . left , newNode )
         else  root . right ← insertRec ( root . right , newNode )
         fi
         Result ← rebalance ( root )  ;  adjustHeight ( Result )
   fi
end
```

Height Pseudocode

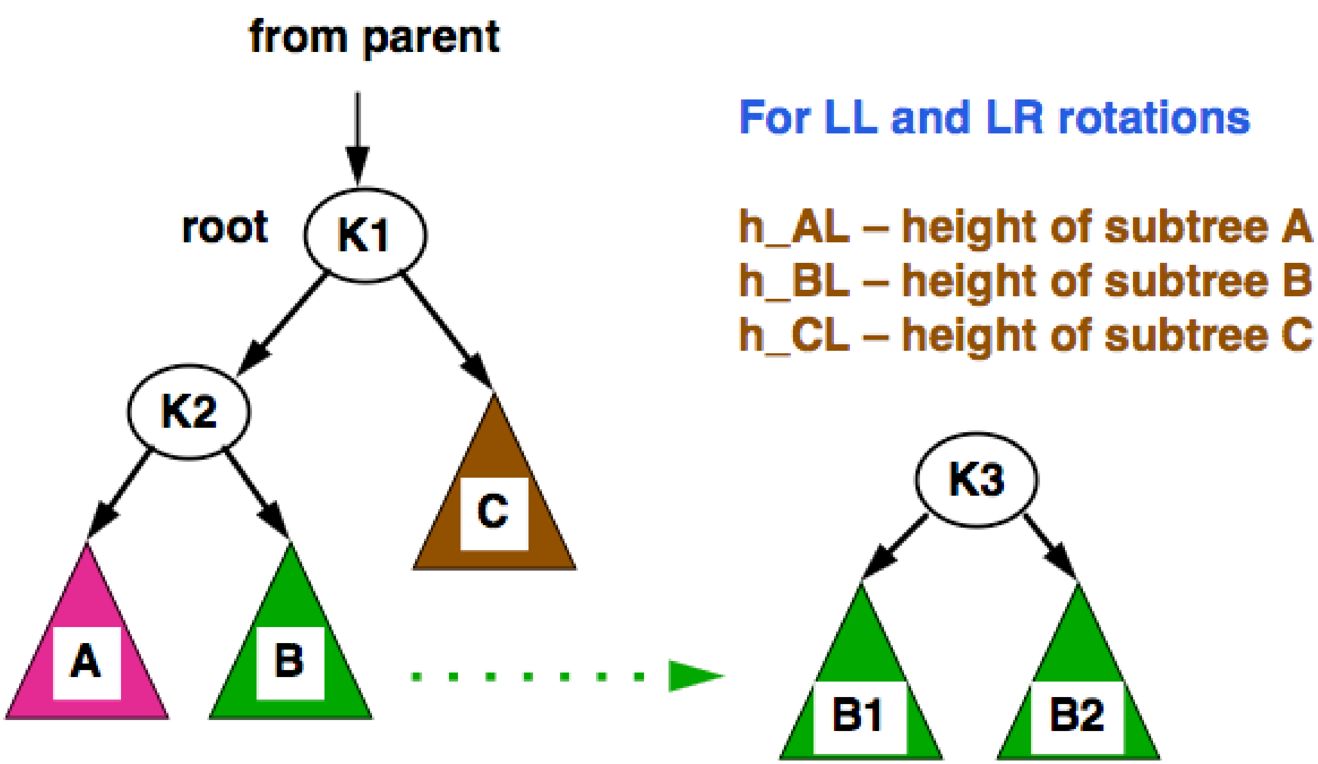
```
// Assume that every node contains a height attribute

// Different definition for height for AVL trees.
// Height of leaf is 1 (Figure 10.10 p435) not 0 (page 273).
// By implication height of empty tree is 0 (see slides
// Tree Algorithms–11..15 on binary tree height).

height ( root : Node ) : Integer is
  if node = Void then Result ← 0
                 else Result ← node . Height
  fi
  return
end
```

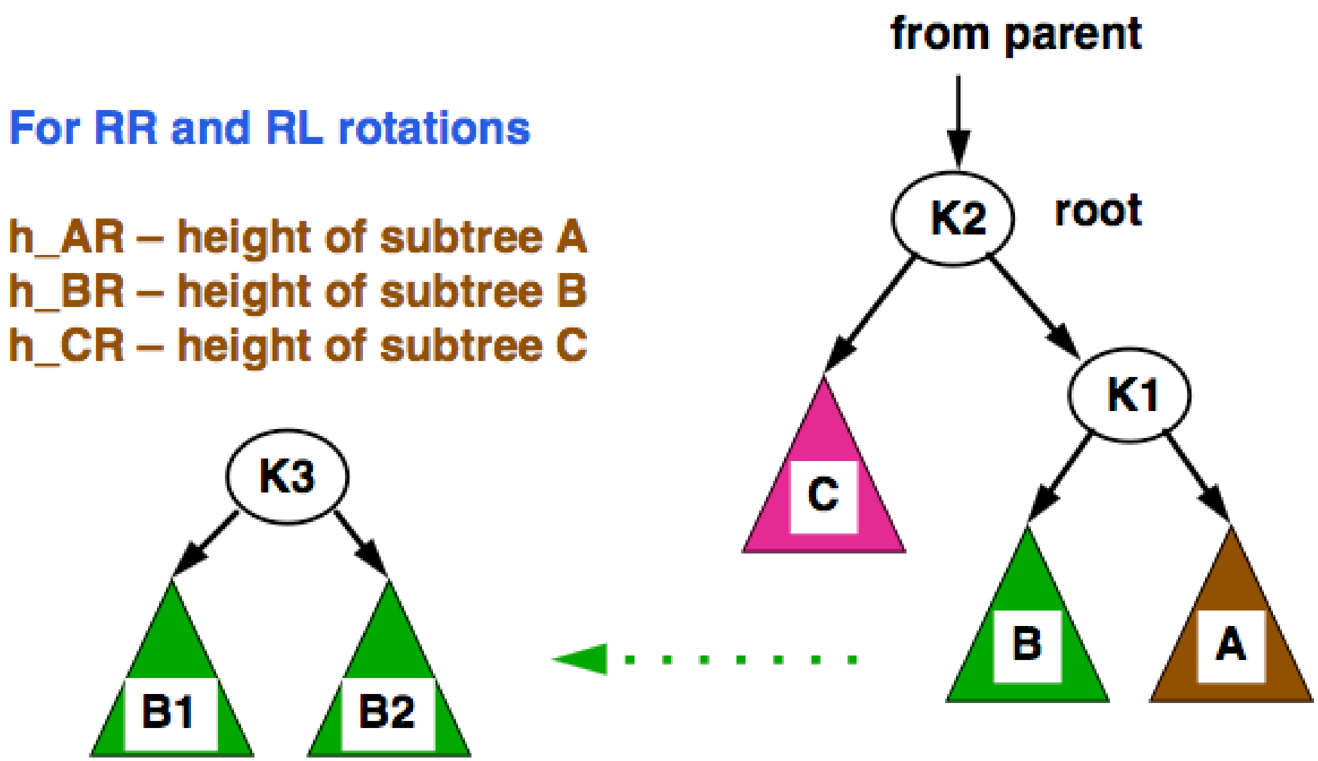
Rebalance Pseudocode

- Define 6 variables that have the height of the sub-trees of interest for rotations
  - » **If any of the pointers are void, height 0 is returned**



Rebalance Pseudocode – 2

- Have the symmetric cases for the other 3 height variables



Rebalance Pseudocode – 3

```
rebalance ( root : Node ) : Node is
  h_AL ← heightLL ( root ) ;  h_AR ← heightRR ( root )
  h_BL ← heightLR ( root ) ;  h_BR ← heightRL ( root )
  h_CL ← height( root . right)  ;  h_CR ← height ( root . left)

  if       h_AL = h_BL  ∧  h_BL ≥ h_CL then Result ← rotate_LL ( root )
  elseif  h_AR = h_BR  ∧  h_BR ≥ h_CR then Result ← rotate_RR ( root )
  elseif  h_BL = h_AL  ∧  h_AL ≥ h_CL then Result ← rotate_LR ( root )
  elseif  h_BR = h_AR  ∧  h_AR ≥ h_CR then Result ← rotate_RL ( root )
  else Result ← root
  fi
end
```

This follows the mathematical development in slides 8, 12, 14 and works correctly for insertion where the objective is to reduce the height of a subtree.   See slides 29..32 for problems with remove.

Remove Difficulties

- Remove has to do two things
  - » **Return the entry corresponding to the key**
  - » **Rebalance the tree**
    - > **Means adjusting the pointers**
    - > **Without a parent pointer, the path from the root to the node is a singly linked list**
    - > **Need to keep track of the parent node of the root of the sub-tree to rebalance to adjust the pointer to the new sub-tree**
    - > **Consequence is every step we have to look one level deeper than BST remove algorithm**
- Rebalancing may occur at all levels

Remove Pseudocode

```
remove ( key : KeyType ) : EntryType is
   if root = Void then Result ← Void     // Entry not in tree
   elseif root . key = key then          // Root is a special case
      Result ←  root . entry
      root ← removeNode ( root )
   else Result ← removeRec ( root , key )  // Try sub-trees
   fi

// The following routines need look ahead.  They are the
// main change from BST remove.

   adjustHeight ( root )
   root ← rebalance ( root )
end
```

RemoveRec Pseudocode

```
// Require  root ≠ null ∧ root .key ≠ key
//          entry ∈ tree → entry ∈ root
//          balanced ( tree (root ) )
// Ensure   entry ∈ tree → Result = entry
//          entry ∉ tree → Result = Void
//          tree ( root ) may be unbalanced

removeRec ( root : Node , key : KeyType ) : EntryType is
   if root . key > key then // Remove from the left sub-tree
   else // Remove from the right sub-tree
   fi
   return
end
```

RemoveRec Pseudocode – 2

```
// Remove from the left sub-tree

   if root . left = Void then Result ← Void
   elseif root . left . key = key then
      Result ← root . left . entry
      root . left ← removeNode ( root . left )
   else
      Result ← removeRec ( root . left , key )
      adjustHeight( root . left )
      root . left ← rebalance ( root . left )
   fi
end
```

RemoveRec Pseudocode – 3

// Remove from the right sub-tree

```
if root . right = Void then Result ← Void
elseif root . right . key = key then
    Result ← root . right . entry
    root . right ← removeNode ( root . right )
else
    Result ← removeRec ( root . right, key )
    adjustHeight ( root . right )
    root . right ← rebalance ( root . right )
fi
end
```

RemoveNode

```
// Require root ≠ Void
// Ensure  Result is a balanced tree with root removed
           Result = replacement node

removeNode ( root : Node ) : Node
   if root . left = Void then Result ← root . right
   elseif root . right = Void then Result ← root . Left
   else child ← root . left
      if child . right = Void then
         root . entry ← child . entry  ;  root . left ← child . left
      else root . left ←
                swap_and_remove_left_neighbour ( root , child )
      fi
      adjustHeight ( root )
      Result ← rebalance ( root )
   fi
end
```

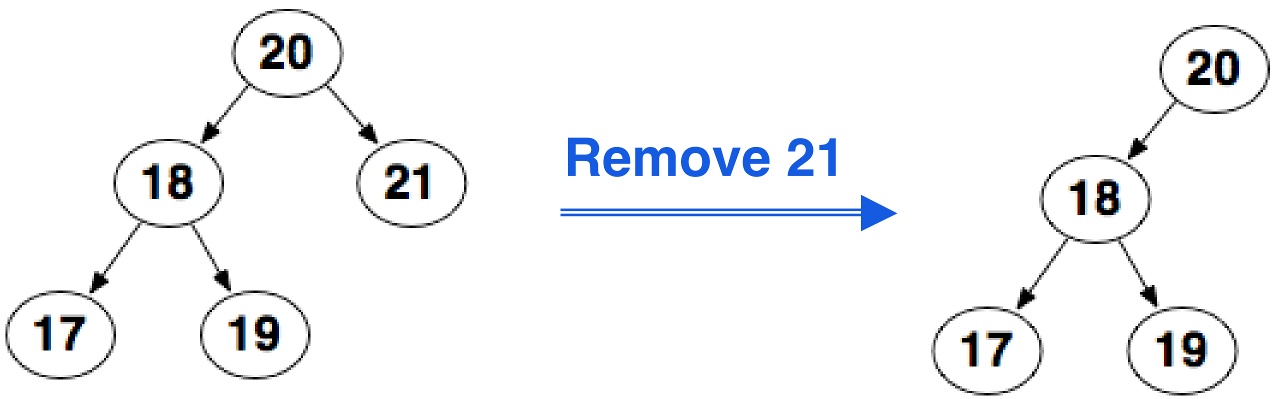
Swap and Remove Left Neighbour

```
// Require child . right ≠ Void
// Ensure Result is a balanced tree with node removed
              Result = replacement node

swap_and_remove_left_neighbour ( parent , child : Node ) : Node
   if child . right . right ≠ Void then
      child . right ←
        swap_and_remove_left_neighbour ( parent , child . right )
   else
      parent . entry ← child . right . entry
      child . right ← child . right . left
   fi
   adjustHeight ( parent )
   Result ← rebalance ( parent )
end
```

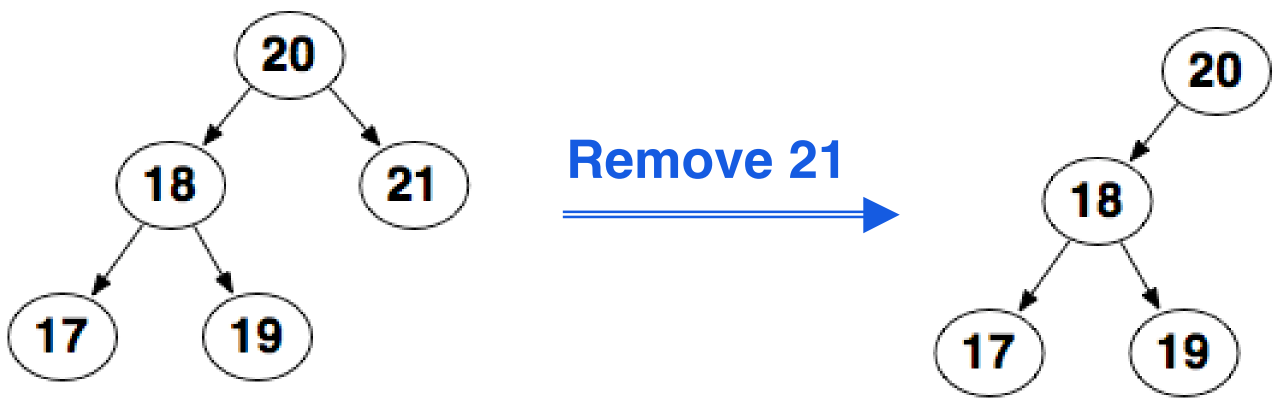
Problem with Rebalance Pseudocode

- The pseudocode for rebalance in slide 21 is works correctly for inserting a node into an AVL tree.
  - » **But the pseudocode fails for the following remove example**



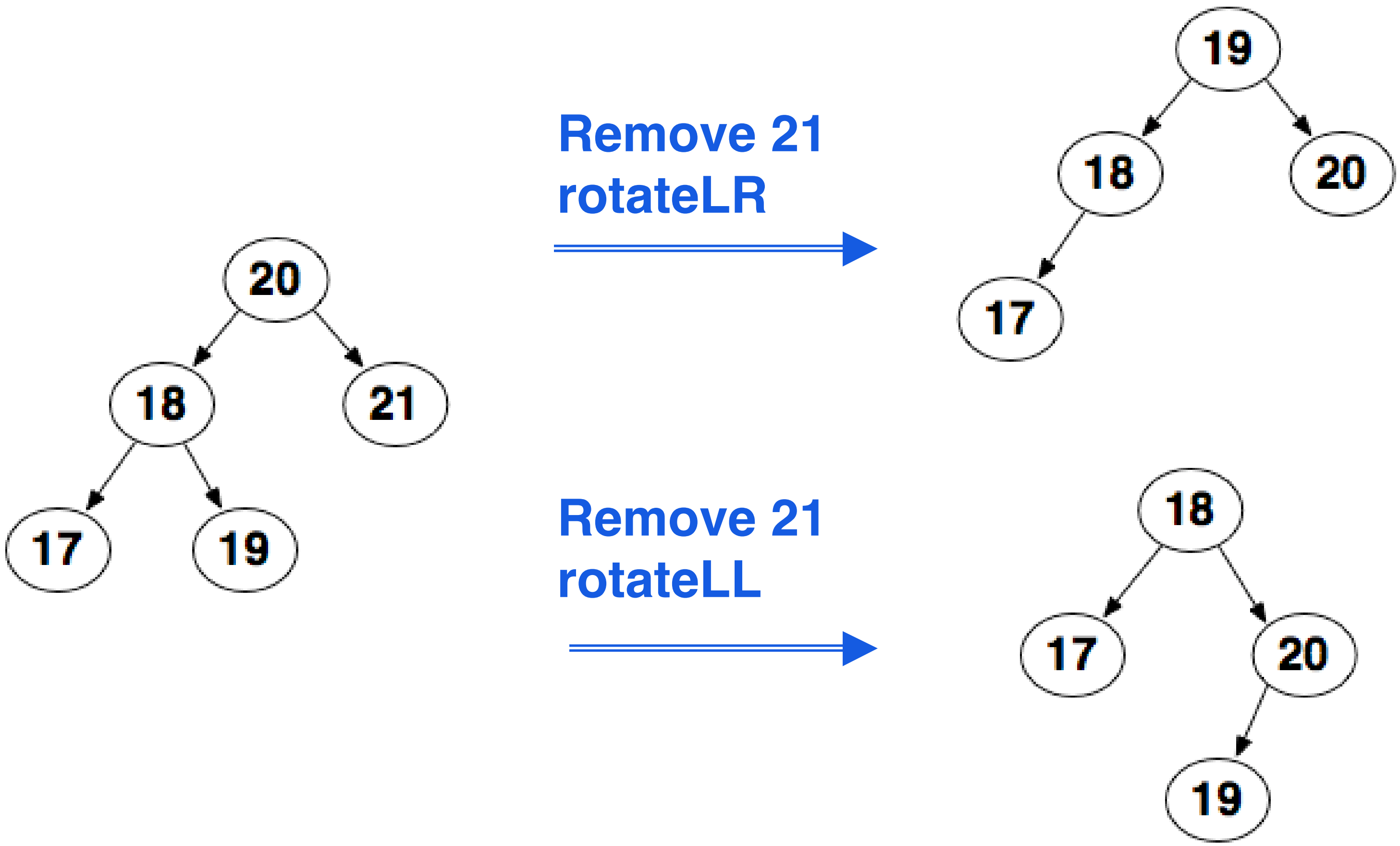
Problem with Rebalance Pseudocode

- What is the problem?
  - » **The case cannot occur on insertion – inserting 17 or 19 invokes a rebalance**
  - » **Need to rebalance but the height will not change**



Rebalance Pseudocode Revised – 2

- Correct removal with rebalance is the following



Rebalance Pseudocode Revised – 3

- Correct rebalance needs to have the following changes
  - » **Does the height of left and right sub-trees differ by more than 1?**
    - > **If so, then continue rebalance.**
  - » **The condition h(A) > h(B) does not hold (slide 8)**
    - > **Need to change to h(A) ≥ h(B)**
      - – **If h(A) = h(B) then either rotateLL or rotateLR will restore balance but not change the height**

Rebalance Pseudocode for Remove

```
rebalance ( root : Node ) : Node is
  h_AL ← heightLL ( root ) ;  h_AR ← heightRR ( root )
  h_BL ← heightLR ( root ) ;  h_BR ← heightRL ( root )
  h_CL ← height( root . right)  ;  h_CR ← height ( root . left)

  if       h_AL ≥ h_BL  ∧  h_BL ≥ h_CL then Result ← rotate_LL ( root )
  elseif  h_AR ≥ h_BR  ∧  h_BR ≥ h_CR then Result ← rotate_RR ( root )
  elseif  h_BL ≥ h_AL  ∧  h_AL ≥ h_CL then Result ← rotate_LR ( root )
  elseif  h_BR ≥ h_AR  ∧  h_AR ≥ h_CR then Result ← rotate_RL ( root )
  else Result ← root
  fi
end
```

Note the ≥ instead of = to handle cases for remove.