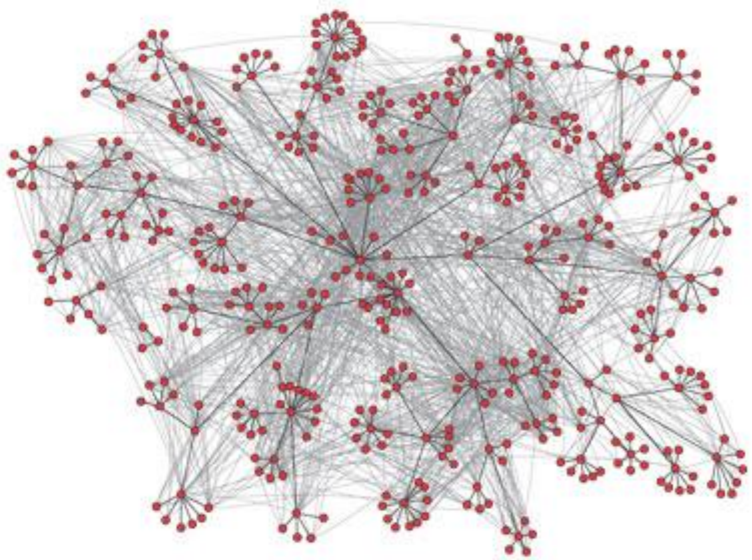
Power-Law Degree Distributions

Thanks to Jure Leskovec, Stanford and Panayiotis Tsaparas, Univ. of Ioannina for slides

Agenda

- Power-law distributions
  - Exponential vs Power-law Distributions
  - Scale-free Networks
  - The anatomy of the long-tail
- Mathematics of Power-laws
- Estimating Power-law Exponent Alpha
- Consequence of Power-Law Degrees

Network Formation Processes

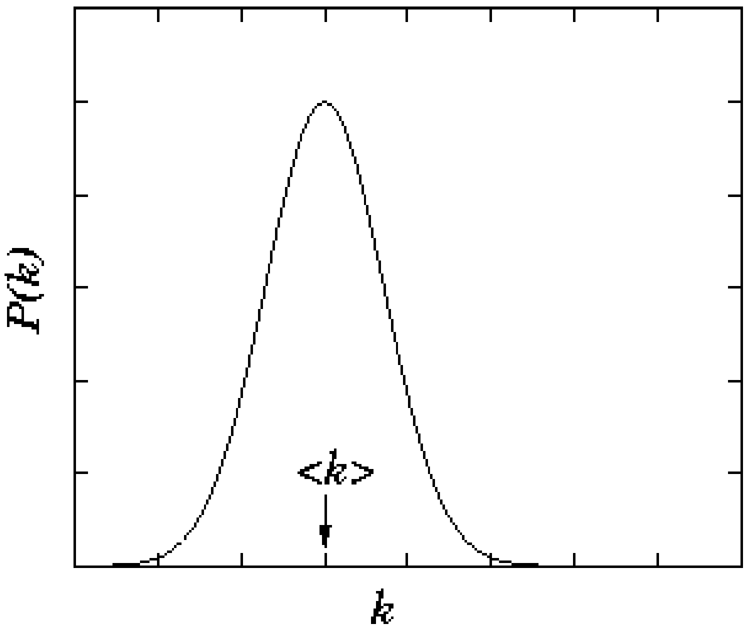


What do we observe that needs explaining

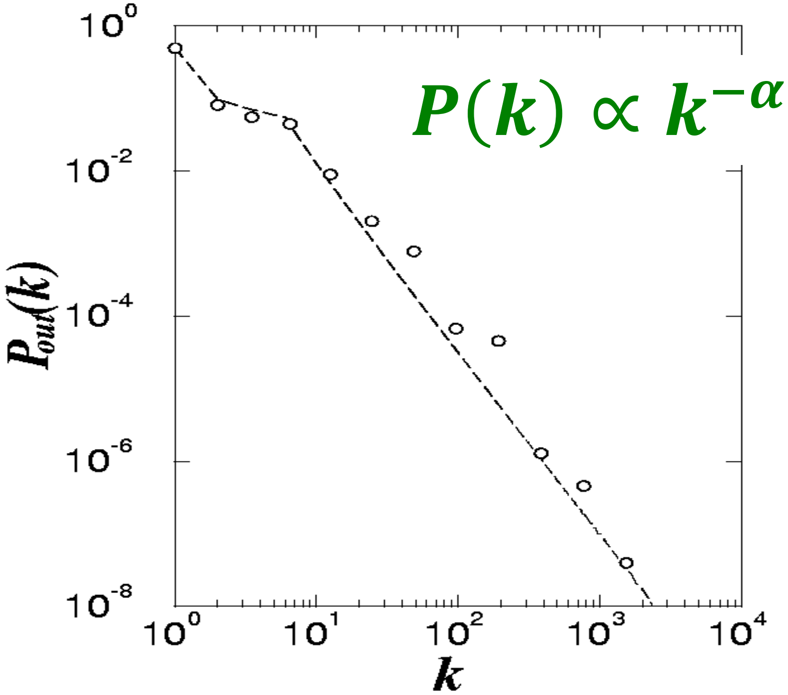
- **Small-world model?**
  - Diameter
  - Clustering coefficient
- **Preferential Attachment:**
  - **Node degree distribution**
    - What fraction of nodes has degree $\boldsymbol{k}$ (as a function of $\boldsymbol{k}$)?
    - Prediction from simple random graph models: $\boldsymbol{p(k)} =$ **exponential function of** $\boldsymbol{k}$
    - **Observation: Often a power-law:** $\boldsymbol{p(k) = k^{-\alpha}}$

Degree Distributions

Expected based on G_{np}



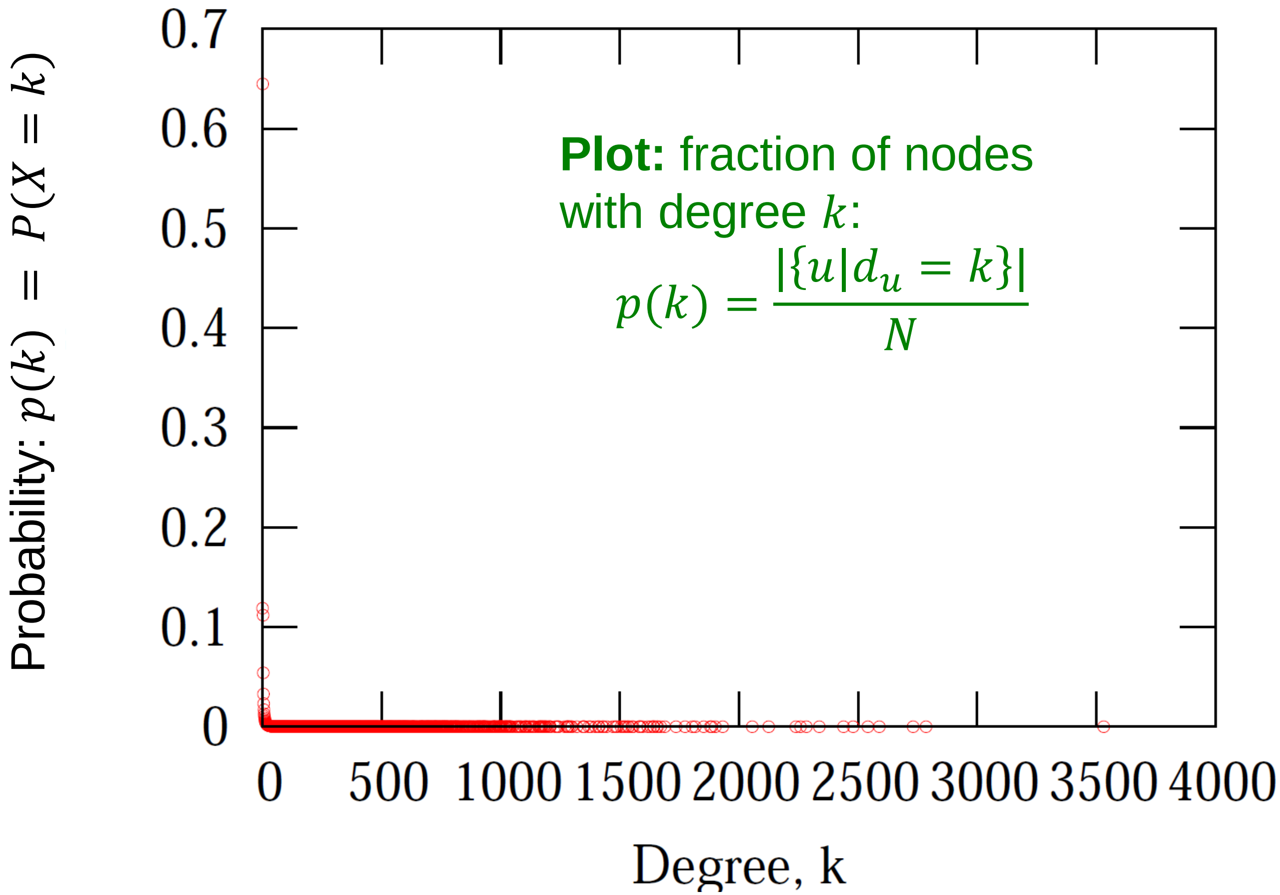
Found in data



$$P(k) \propto k^{-\alpha}$$

Node Degrees in Networks

- **Take a network, plot a histogram of $P(k)$ vs. k**

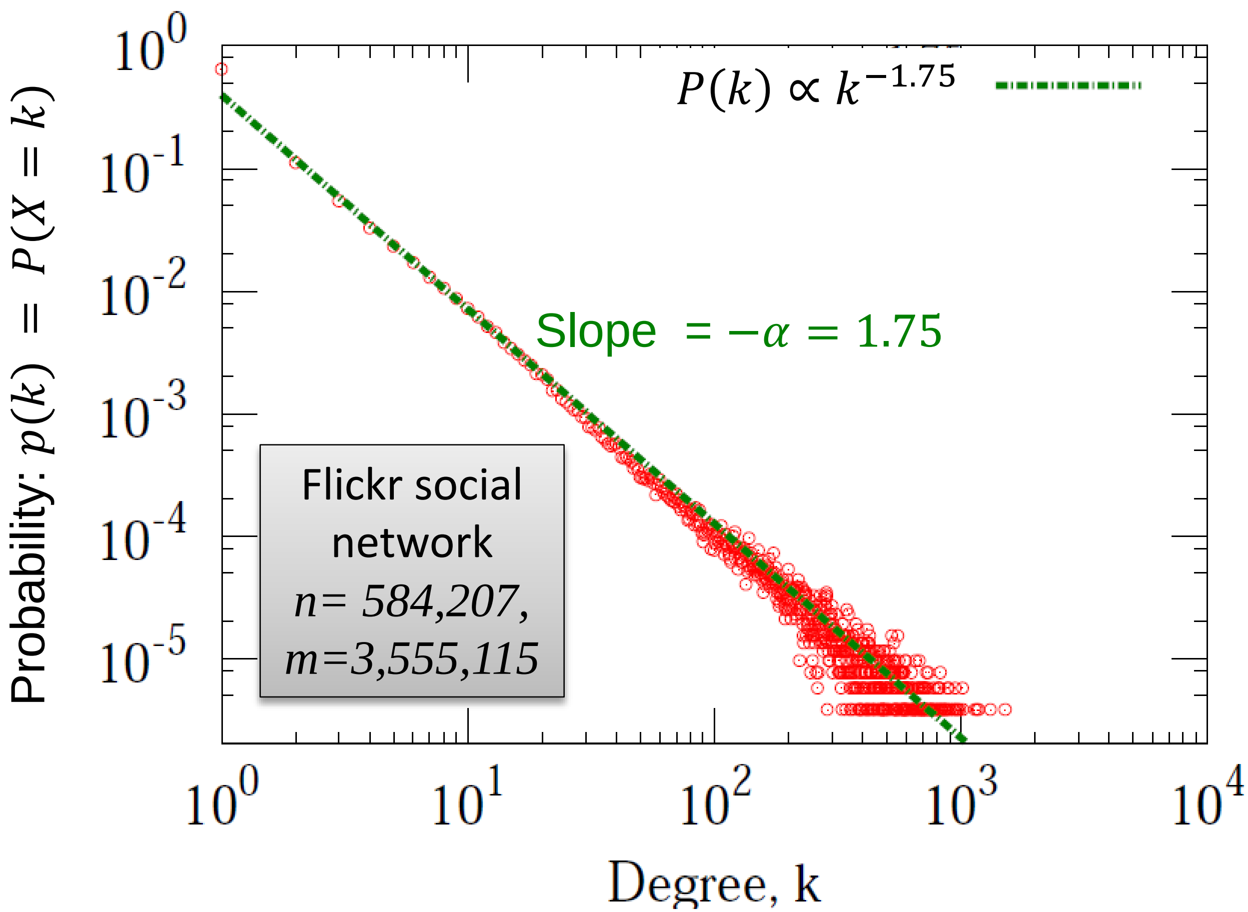


Flickr social network
n= 584,207,
m=3,555,115

[Leskovec et al. KDD '08]

Node Degrees in Networks

- **Plot the same data on *log-log* scale:**

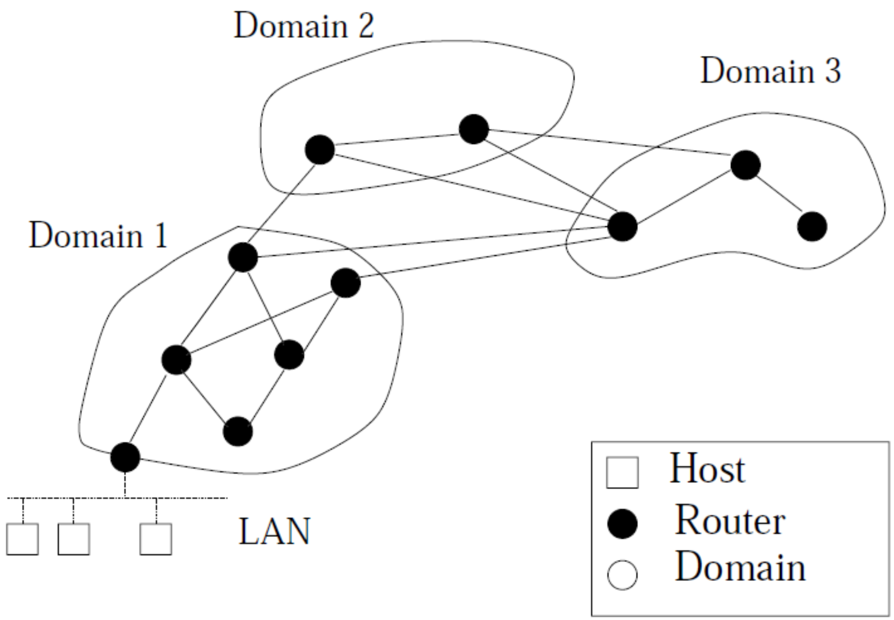


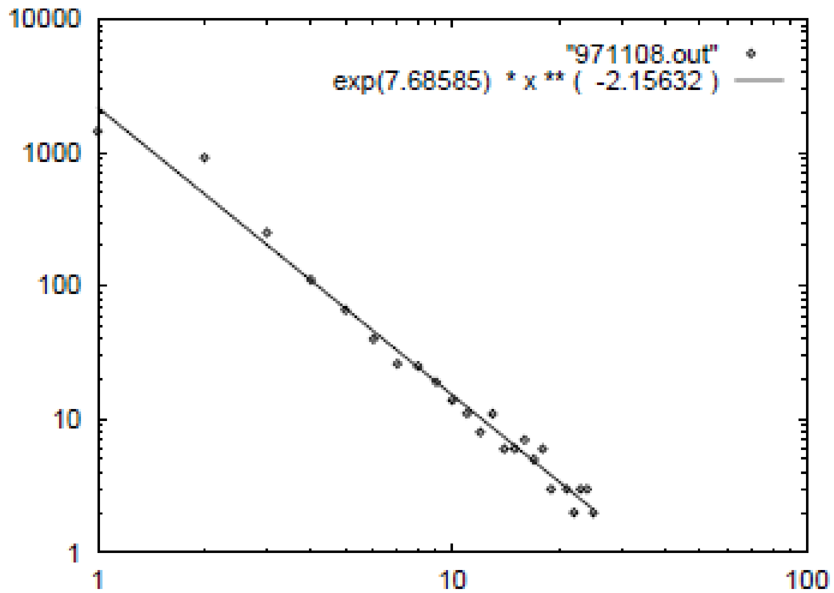
How to distinguish:
$P(k) \propto \exp(-k)$ **vs.**
$P(k) \propto k^{-\alpha}$ **?**

Take logarithms:
if $y = f(x) = e^{-x}$ then
$\log(y) = -x$
If $y = x^{-\alpha}$ then
$\log(y) = -\alpha \log(x)$
So, on log-log axis power-law looks like a straight line of slope $-\alpha$!

Node Degrees: Faloutsos[3]

- **Internet Autonomous Systems** [Faloutsos, Faloutsos and Faloutsos, 1999]

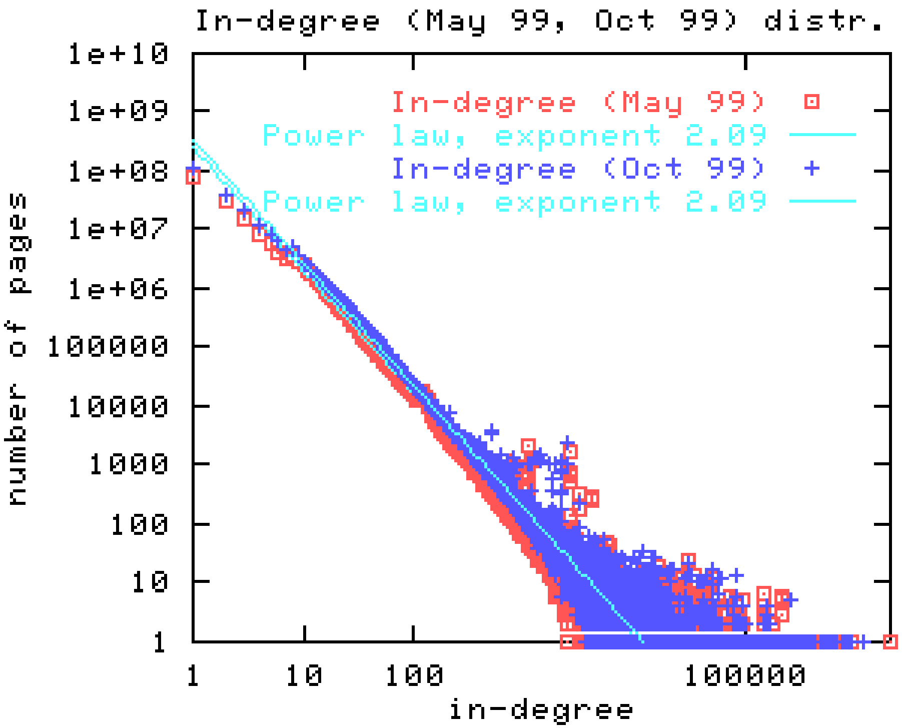


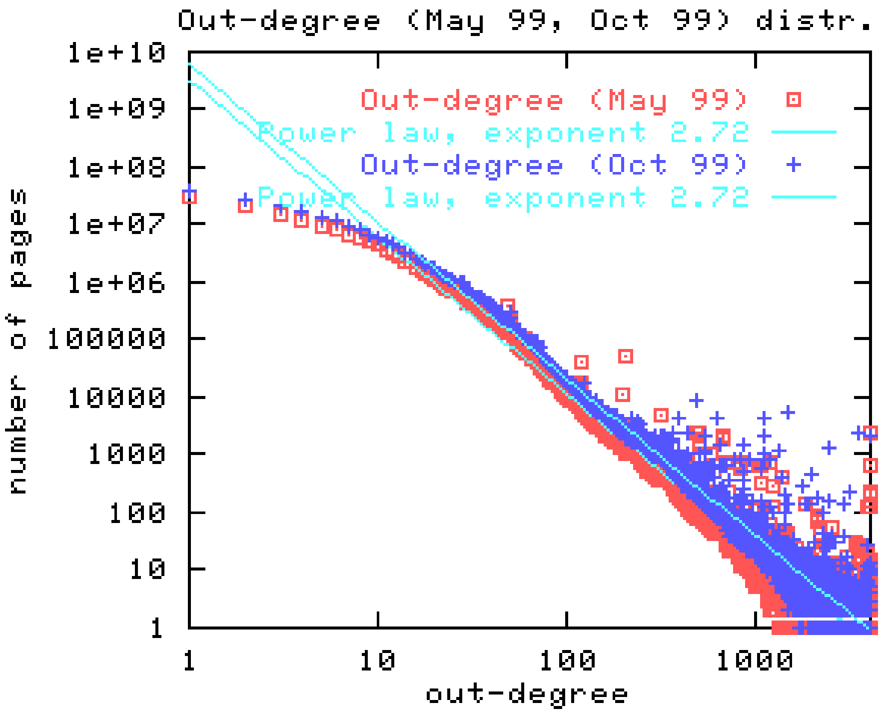


Internet domain topology

Node Degrees: Web

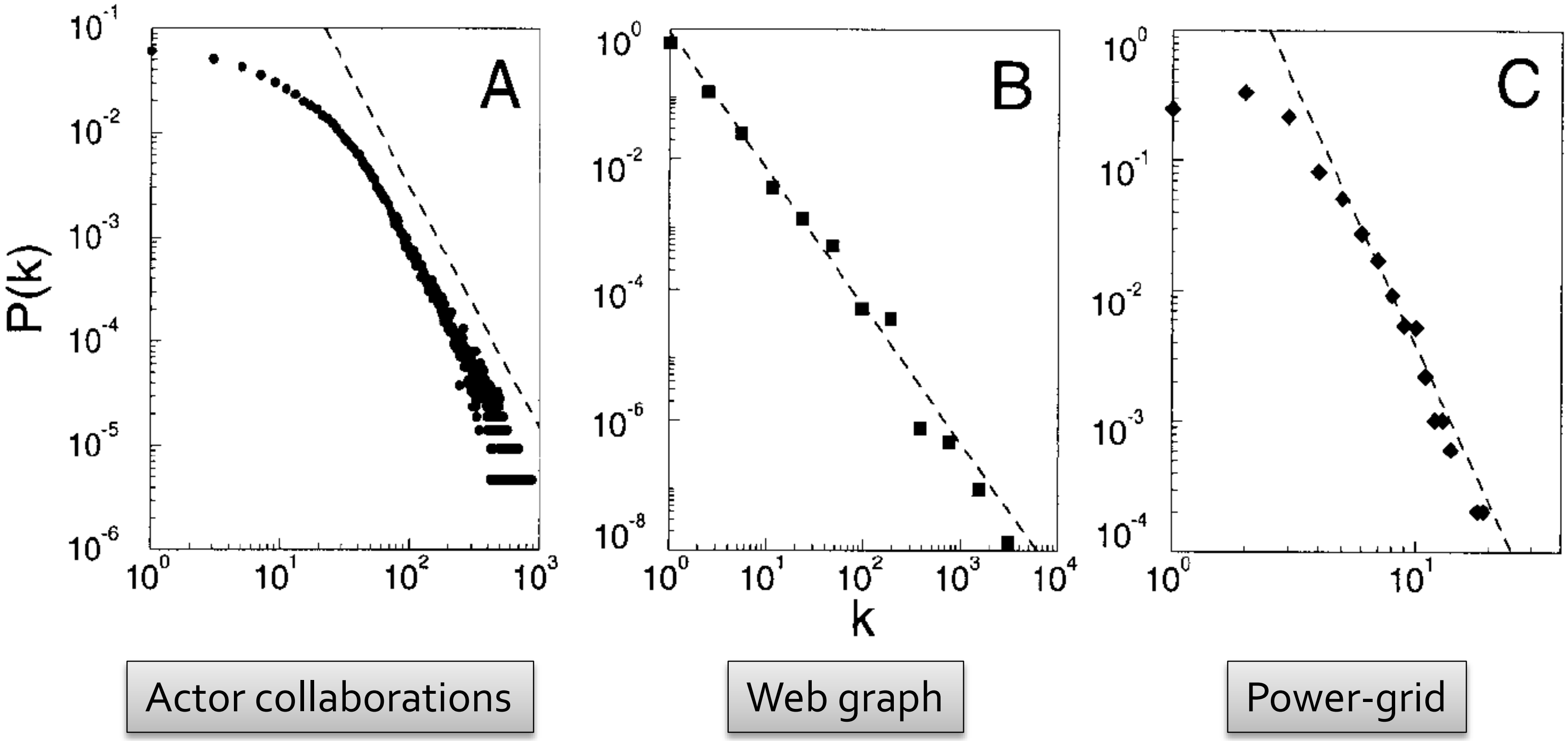
- **The World Wide Web** [Broder et al., 2000]



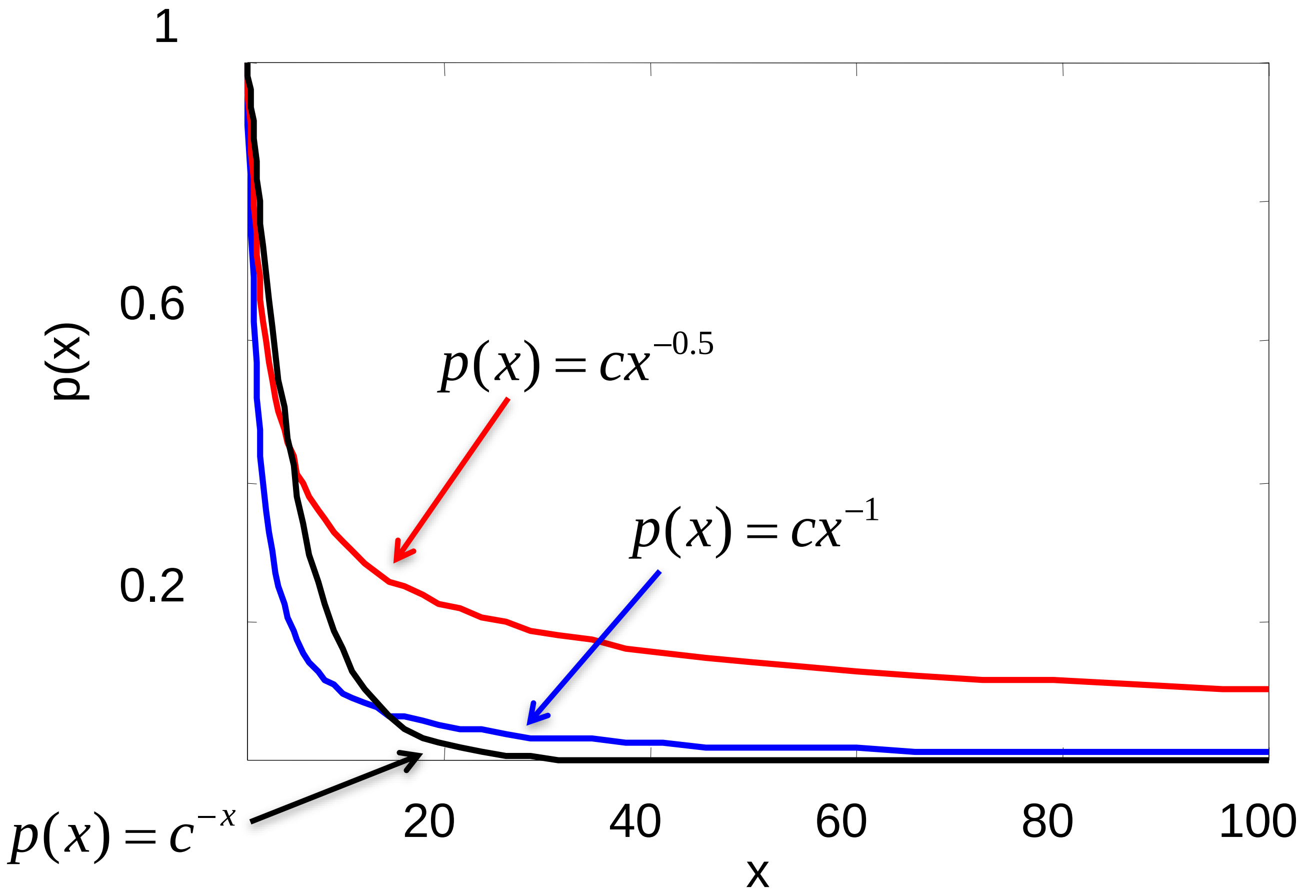


Node Degrees: Barabasi&Albert

- **Other Networks** [Barabasi-Albert, 1999]



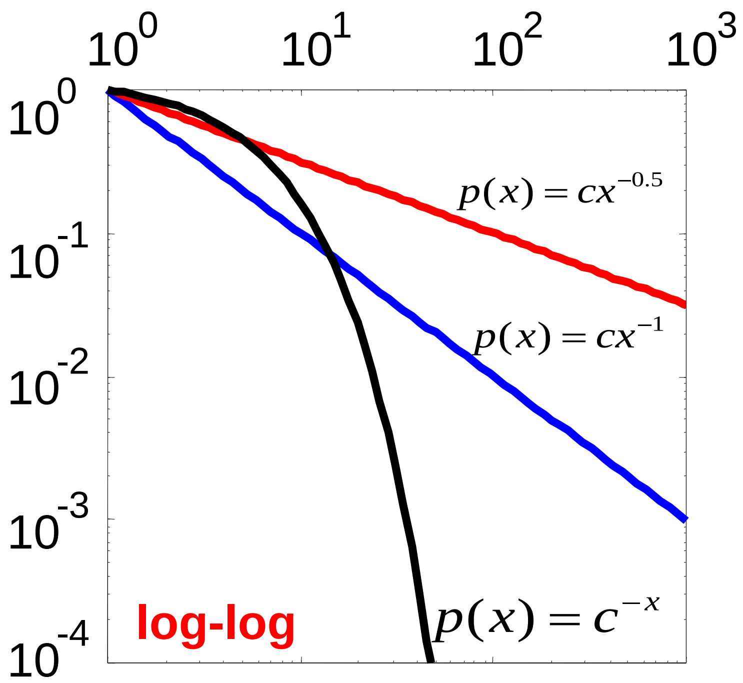
Exponential vs. Power-Law

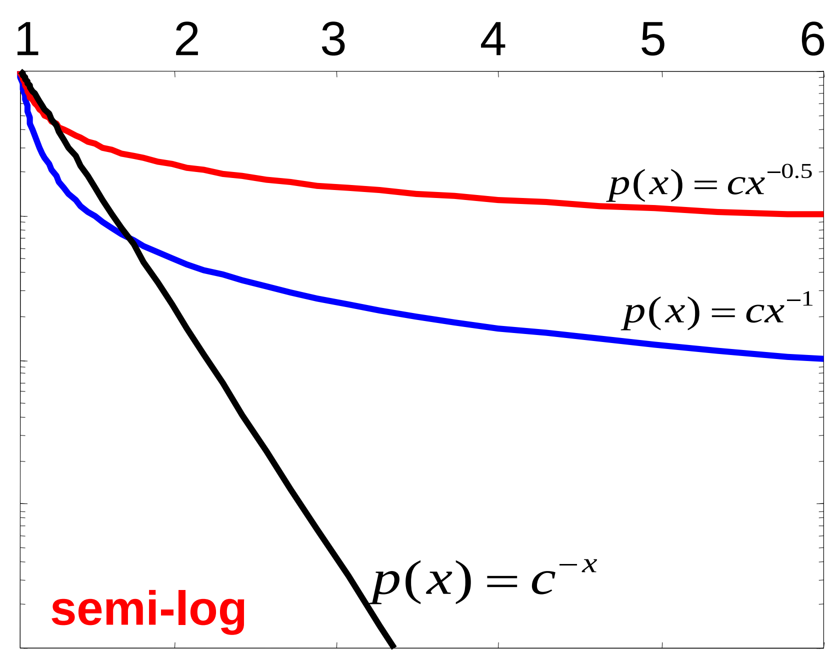


- Above a certain x value, the power law is always higher than the exponential!

Exponential vs. Power-Law

- **Power-law vs. Exponential on log-log and semi-log (log-lin) scales**

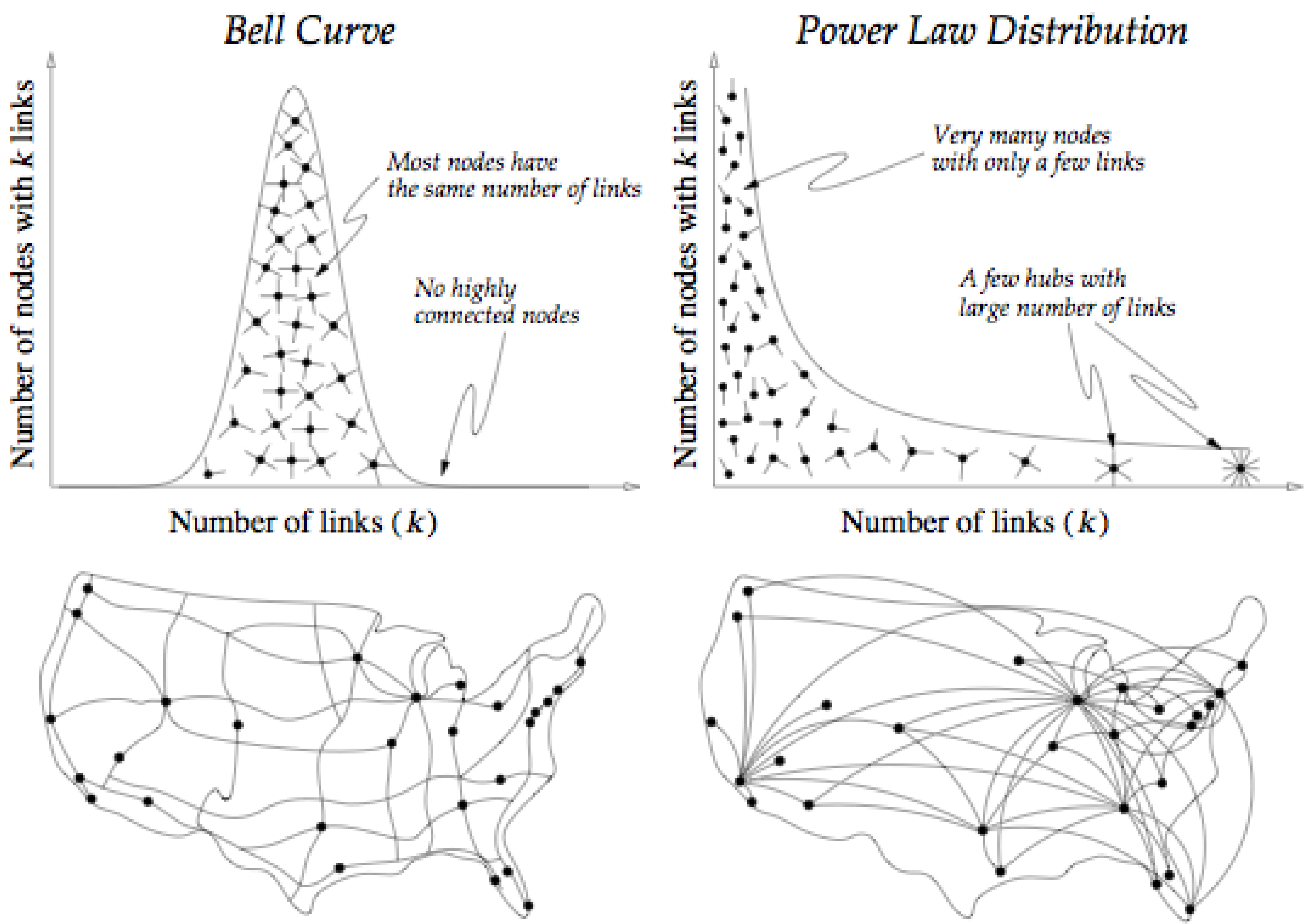




x … logarithmic axis
y … logarithmic axis

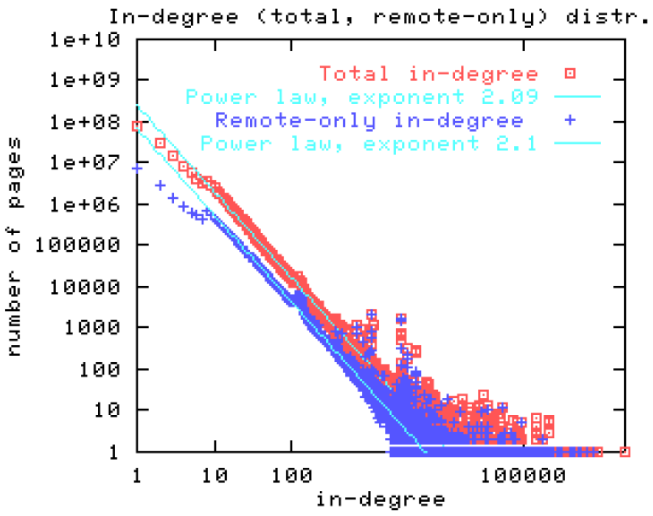
x … linear
y … logarithmic

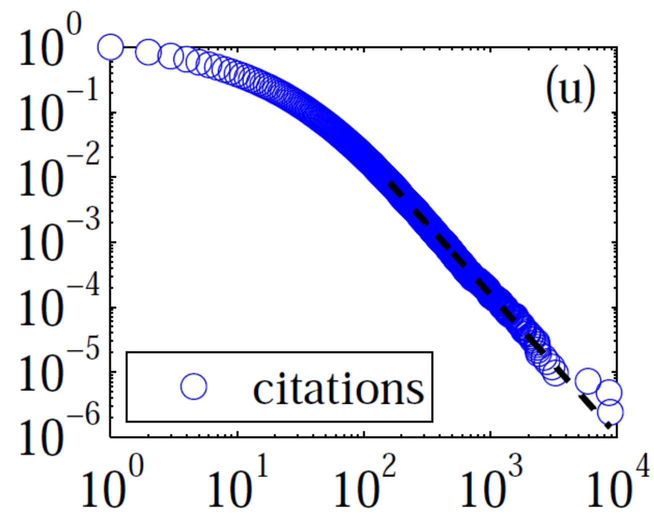
Exponential vs. Power-Law



Power-Law Degree Exponents

- **Power-law degree exponent is typically $2 < \alpha < 3$**
  - **Web graph:**
    - $\alpha_{in} = 2.1$, $\alpha_{out} = 2.4$ [Broder et al. 00]
  - **Autonomous systems:**
    - $\alpha = 2.4$ [Faloutsos^3, 99]
  - **Actor-collaborations:**
    - $\alpha = 2.3$ [Barabasi-Albert 00]
  - **Citations to papers:**
    - $\alpha \approx 3$ [Redner 98]
  - **Online social networks:**
    - $\alpha \approx 2$ [Leskovec et al. 07]





Scale-Free Networks

- **Definition:**
  Networks with a power-law tail in their degree distribution are called "scale-free networks"

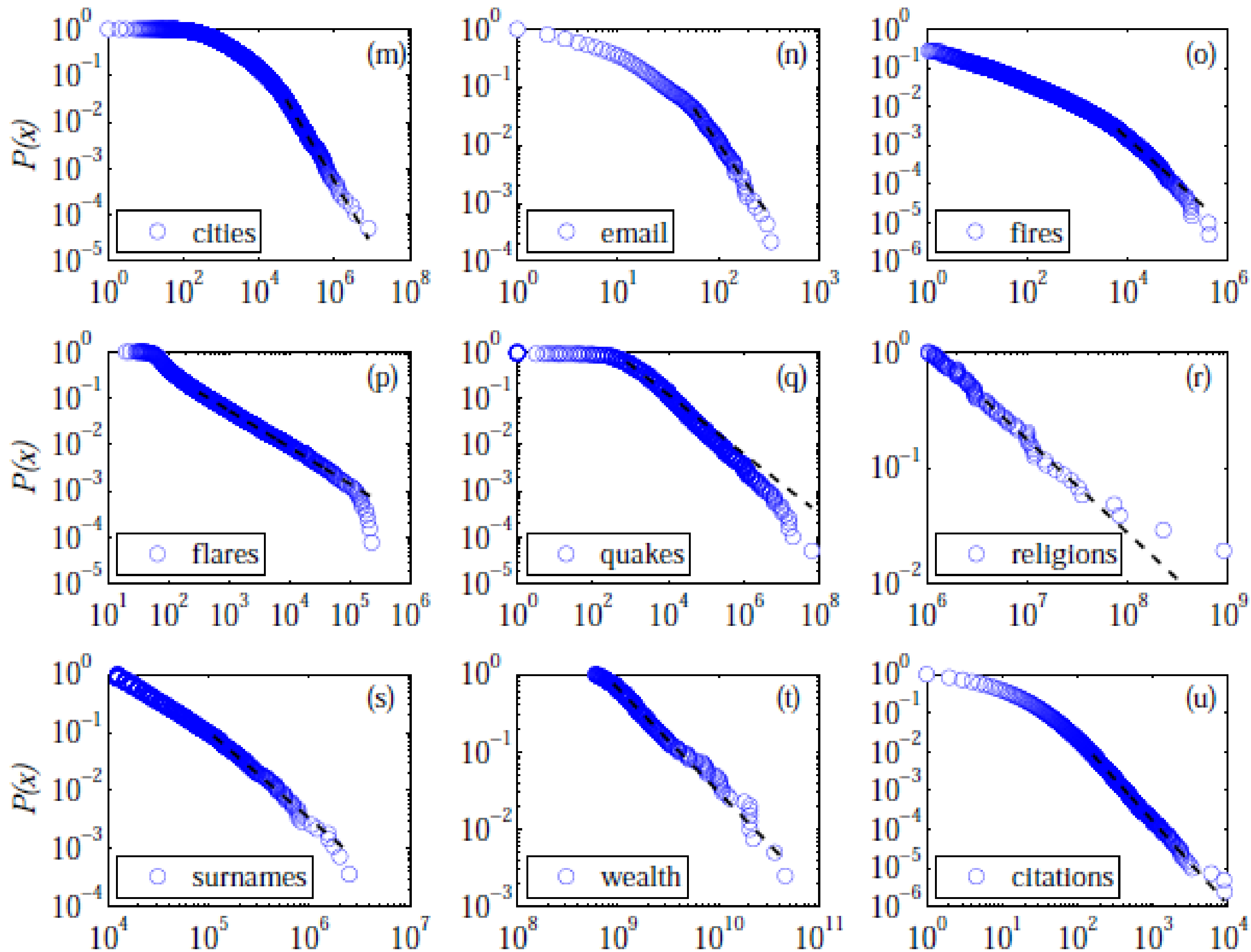
- **Where does the name come from?**
  - **Scale invariance:** There is no characteristic scale
  - **Scale-free function:** $\boldsymbol{f(ax) = a^{\lambda} f(x)}$
    - Power-law function: $\boldsymbol{f(ax) = a^{\lambda} x^{\lambda} = a^{\lambda} f(x)}$

Log() or Exp() are not scale free!

$f(ax) = \log(ax) = \log(a) + \log(x) = \log(a) + f(x)$

$f(ax) = \exp(ax) = \exp(x)^{a} = f(x)^{a}$

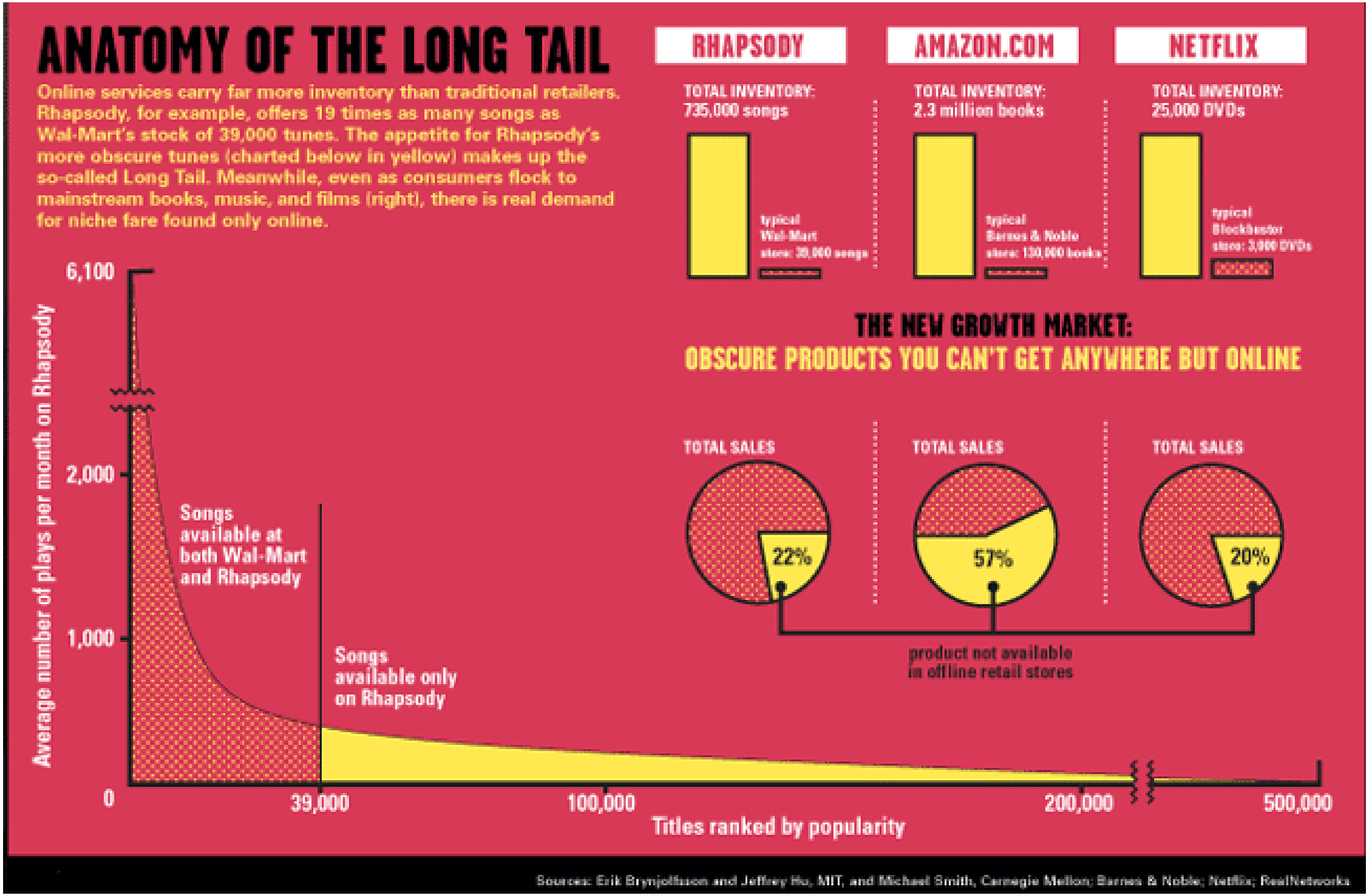
Power-Laws are Everywhere



Many other quantities follow heavy-tailed distributions

[Chris Anderson, Wired, 2004]

Anatomy of the Long Tail



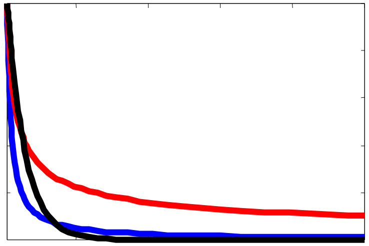
Mathematics of Power-Laws

Heavy Tailed Distributions

- **Degrees are heavily skewed:**

  Distribution $P(X > x)$ is **heavy tailed if:**

$$\lim_{x \to \infty} \frac{P(X > x)}{e^{-\lambda x}} = \infty$$



- **Note:**
  - **Normal PDF:** $p(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$
  - **Exponential PDF:** $p(x) = \lambda e^{-\lambda x}$
    - **then** $P(X > x) = 1 - P(X \le x) = e^{-\lambda x}$

are not heavy tailed!

[Clauset-Shalizi-Newman 2007]

Heavy Tailed Distributions

- **Various names, kinds and forms:**
  - Long tail, Heavy tail, Zipf's law, Pareto's law
- **Heavy tailed distributions:**
  - **P(x) is proportional to:**

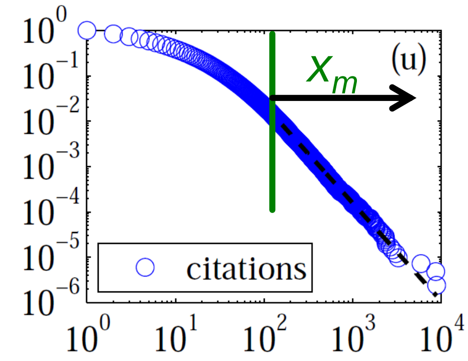
power law	$P(x) \propto x^{-\alpha}$
power law with cutoff	$x^{-\alpha} e^{-\lambda x}$
stretched exponential	$x^{\beta-1} e^{-\lambda x^{\beta}}$
log-normal	$\frac{1}{x} \exp\left[-\frac{(\ln x-\mu)^2}{2\sigma^2}\right]$

Mathematics of Power-laws

- **What is the normalizing constant?**

  $p(x) = Z\, x^{-\alpha}$ $\qquad Z = ?$

  - $\boldsymbol{p(x)}$ is a distribution: $\int \boldsymbol{p(x)dx = 1}$



p(x) diverges as x→0 so x_m is the minimum value of the power-law distribution $x \in [x_m, \infty]$

Continuous approximation

- $1 = \int_{x_m}^{\infty} p(x)dx = Z \int_{x_m}^{\infty} x^{-\alpha} dx$
- $= -\frac{Z}{\alpha-1}\left[x^{-\alpha+1}\right]_{x_m}^{\infty} = -\frac{Z}{\alpha-1}\left[\infty^{1-\alpha} - {x_m}^{1-\alpha}\right]$

Need: **α > 1 !**

- $\Rightarrow Z = (\alpha - 1)x_m^{\alpha-1}$

$$\boldsymbol{p(x) = \frac{\alpha - 1}{x_m}\left(\frac{x}{x_m}\right)^{-\alpha}}$$

Integral:

$$\int (ax)^n = \frac{(ax)^{n+1}}{a(n+1)}$$

[Clauset-Shalizi-Newman 2007]

Mathematics of Power-laws

- **What's the expected value of a power-law random variable *X*?**
- $E[X] = \int_{x_m}^{\infty} x\, p(x) dx = Z \int_{x_m}^{\infty} x^{-\alpha+1} dx$
- $= \frac{Z}{2-\alpha}\left[x^{2-\alpha}\right]_{x_m}^{\infty} = \frac{(\alpha-1)x_m^{\alpha-1}}{-(\alpha-2)}\left[\infty^{2-\alpha} - x_m^{2-\alpha}\right]$

Need: $\alpha > 2$!

$$\Rightarrow E[X] = \frac{\alpha - 1}{\alpha - 2} x_m$$

Power-law density:

$$p(x) = \frac{\alpha - 1}{x_m}\left(\frac{x}{x_m}\right)^{-\alpha}$$

$$Z = \frac{\alpha - 1}{x_m^{1-\alpha}}$$

Mathematics of Power-Laws

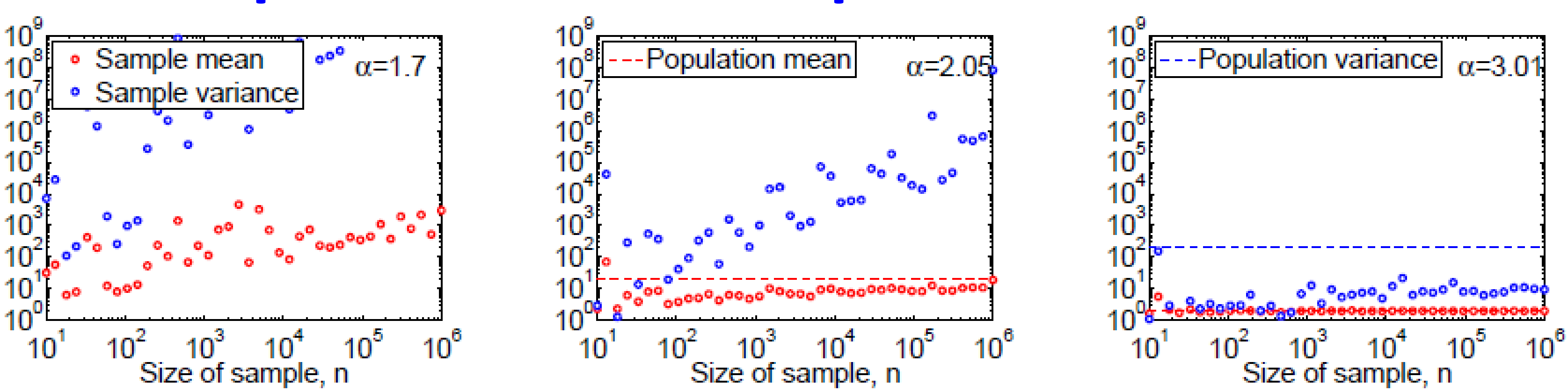
- **Power-laws have infinite moments!**

$$E[X] = \frac{\alpha - 1}{\alpha - 2} x_m$$

  - If $\alpha \leq 2 : E[X] = \infty$
  - If $\alpha \leq 3 : Var[X] = \infty$
    - **Average is meaningless, as the variance is too high!**

In real networks
2 < α < 3 so:
E[*X*] = const
Var[*X*] = ∞

- **Consequence: Sample average of *n* samples from a power-law with exponent α**



Mathematics of Power-Laws

Power-law density:
$$p(x) = \frac{\alpha - 1}{x_m}\left(\frac{x}{x_m}\right)^{-\alpha}$$

- **How to generate power-law distributed random numbers?**
  - We want to generate $\boldsymbol{x}$ … a power-law distributed random number. **Density:** $p(x) = \frac{\alpha-1}{x_m}\left(\frac{x}{x_m}\right)^{-\alpha}$
  - But we have access to $\boldsymbol{r}$ … uniform random number **Density:** $p(r) = 1$ **if** $0 \le r \le 1$ **else** $p(r) = 0$
  - **We want to transform the densities!**
  - $\boldsymbol{P(X \le x) = P(R \le r)}$

Mathematics of Power-Laws

- **Want to generate power-law random numbers!**
  - **Equate the densities:** $P(X \le x) = P(R \le r)$
  - $P(R \le r) = \int_0^r 1 dq = r$
  - $P(X \le x) = \int_{x_m}^{x} \frac{\alpha-1}{x_m}\left(\frac{y}{x_m}\right)^{-\alpha} dy = \left[\frac{\alpha-1}{x_m} \cdot \frac{x_m}{1-\alpha}\left(\frac{y}{x_m}\right)^{1-\alpha}\right]_{x_m}^{x}$

    $= \left[-\left(\frac{y}{x_m}\right)^{1-\alpha}\right]_{x_m}^{x} = -\left(\frac{x}{x_m}\right)^{1-\alpha} + 1$
  - **Putting it all together:** $r = 1 - \left(\frac{x}{x_m}\right)^{1-\alpha}$
  - **Solving for x:** $x = x_m(1-r)^{-1/(\alpha-1)}$

Integral:

$$\int (ax)^n = \frac{(ax)^{n+1}}{a(n+1)}$$

Power-law density:

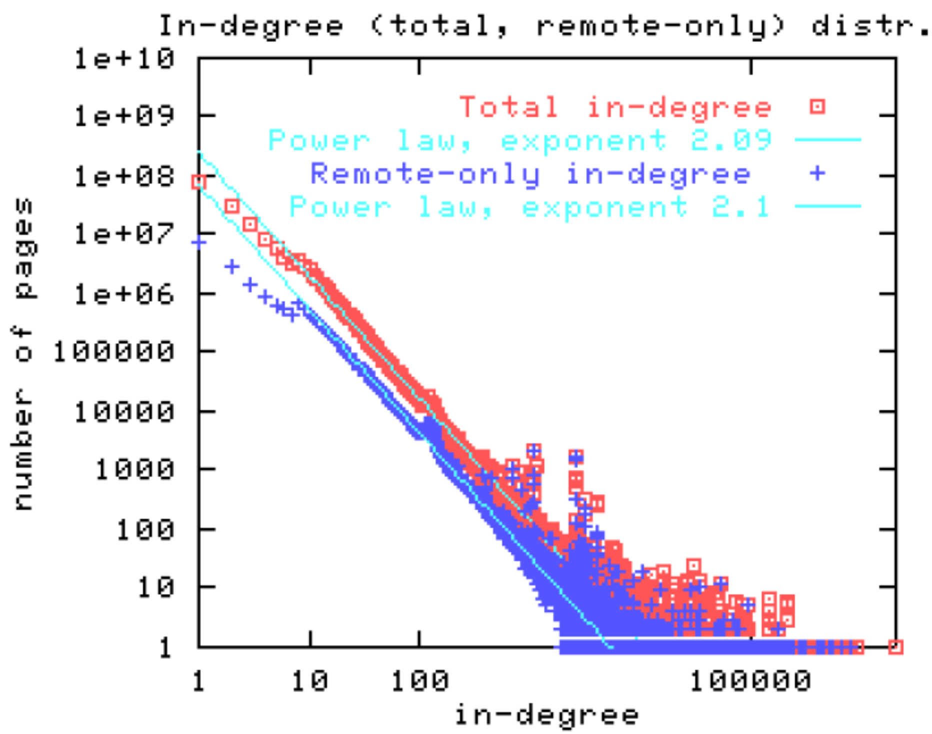
$$p(x) = \frac{\alpha-1}{x_m}\left(\frac{x}{x_m}\right)^{-\alpha}$$

Estimating Power-Law Exponent Alpha

Estimating Power-Law Exponent α

Estimating α from data:

- **(1)** Fit a line on log-log axis using least squares:
  - **Solve** $\arg\min_{\alpha} (\log(y) - \alpha \log(x) + b)^2$



BAD!

Estimating Power-Law Exponent α

OK!

Estimating α from data:

- Plot **Complementary CDF (CCDF)** $P(X \geq x)$. Then the estimated $\boldsymbol{\alpha = 1 + \alpha'}$ where $\boldsymbol{\alpha'}$ is the slope of $P(X \geq x)$.

- <u>**Fact:**</u> **If $\mathbf{p(x) = P(X = x)} \propto x^{-\alpha}$ then $\mathbf{P(X \geq x)} \propto x^{-(\alpha-1)}$**
  - $P(X \geq x) = \sum_{j=x}^{\infty} p(j) \approx \int_x^{\infty} Z\, y^{-\alpha} dy =$
  - $= \frac{Z}{1-\alpha}\left[y^{1-\alpha}\right]_x^{\infty} = \frac{Z}{1-\alpha} - x^{-(\alpha-1)}$

Estimating Power-Law Exponent α

OK!

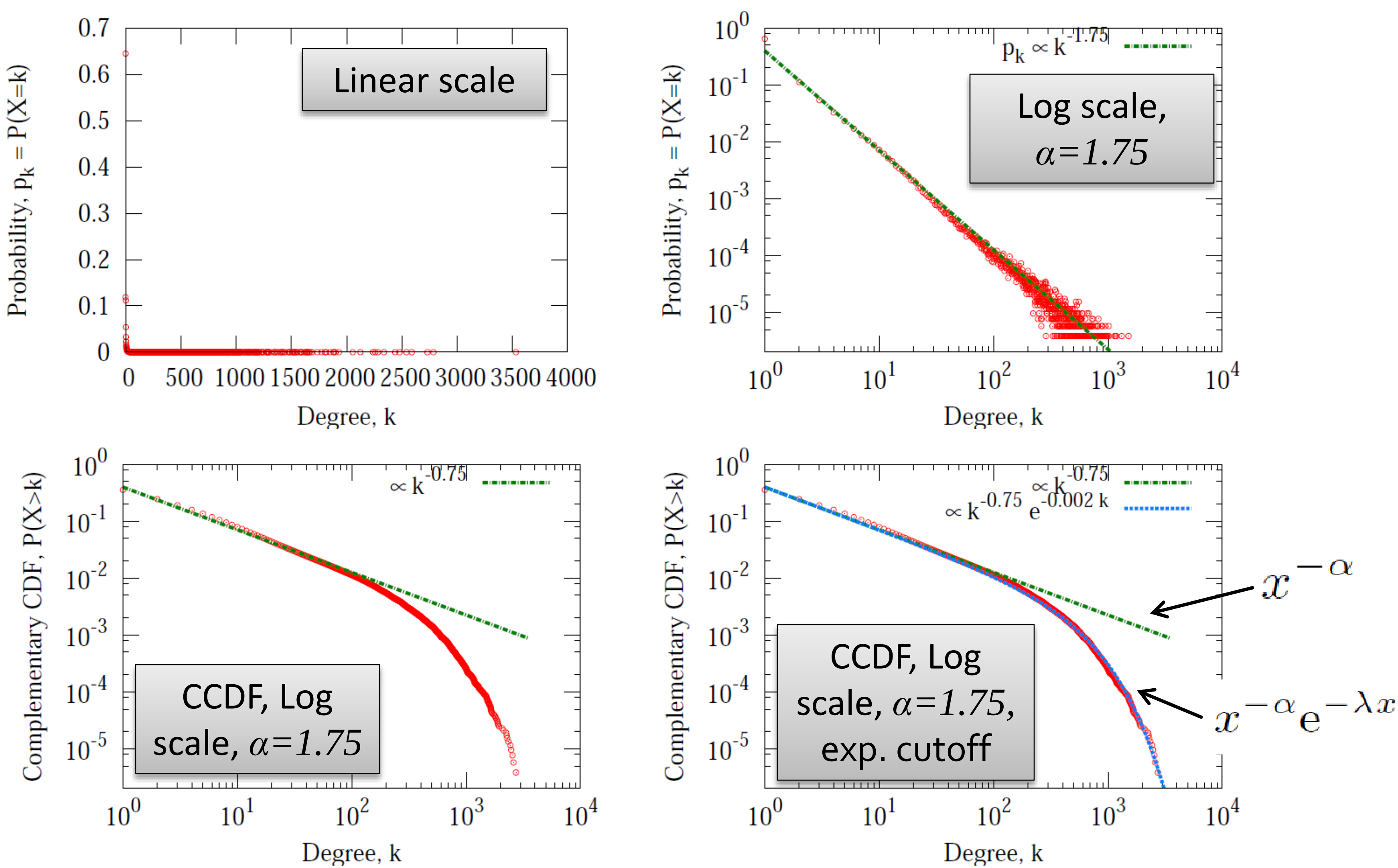
Estimating α from data:

- **Use maximum likelihood approach:**
  - **The log-likelihood of observed data d_i:**
    - $L(\alpha) = \ln\left(\prod_i^n p(d_i)\right) = \sum_i^n \ln p(d_i)$
    - $= \sum_i^n \left(\ln(\alpha - 1) - \ln(x_m) - \alpha \ln\left(\frac{d_i}{x_m}\right)\right)$
  - **Want to find α that max $L(\alpha)$: Set** $\frac{\mathrm{dL}(\alpha)}{\mathrm{d}\alpha} = 0$
  - $\frac{\mathrm{dL}(\alpha)}{\mathrm{d}\alpha} = 0 \;\Rightarrow\; \frac{n}{\alpha - 1} - \sum_i^n \ln\left(\frac{d_i}{x_m}\right) = 0$
  - $\Rightarrow \hat{\alpha} = 1 + n\left[\sum_i^n \ln\left(\frac{d_i}{x_m}\right)\right]^{-1}$

Power-law density:

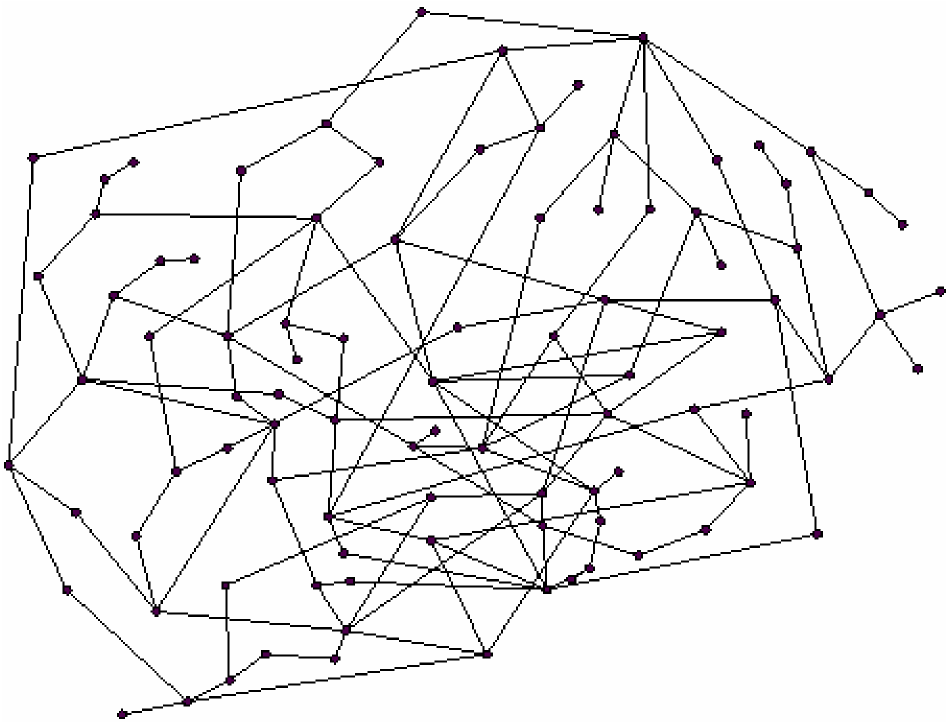
$$p(x) = \frac{\alpha - 1}{x_m}\left(\frac{x}{x_m}\right)^{-\alpha}$$

Flickr: Fitting Degree Exponent



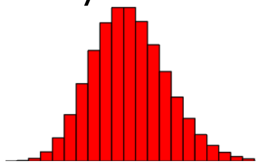
Consequence of Power-Law Degrees

Random vs. Scale-free network

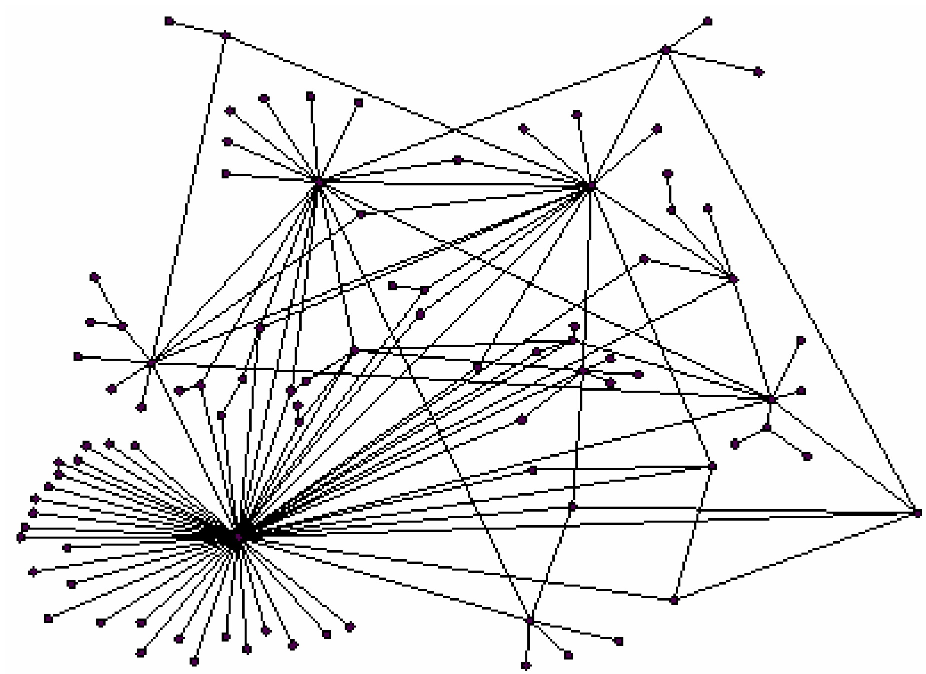


Random network

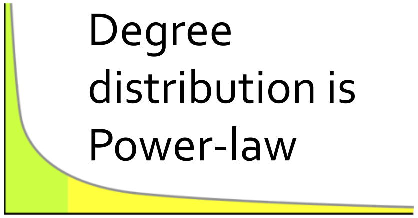
(Erdos-Renyi random graph)



Degree distribution is Binomial



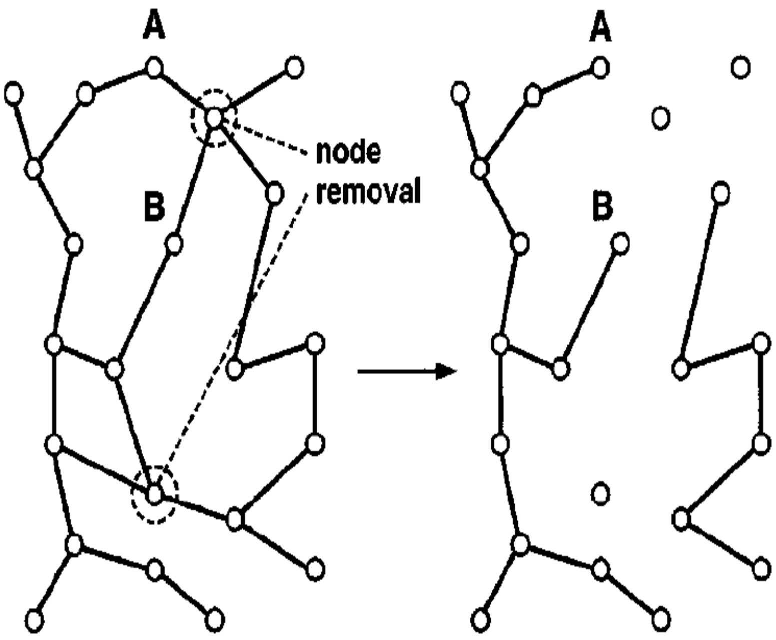
Scale-free (power-law) network



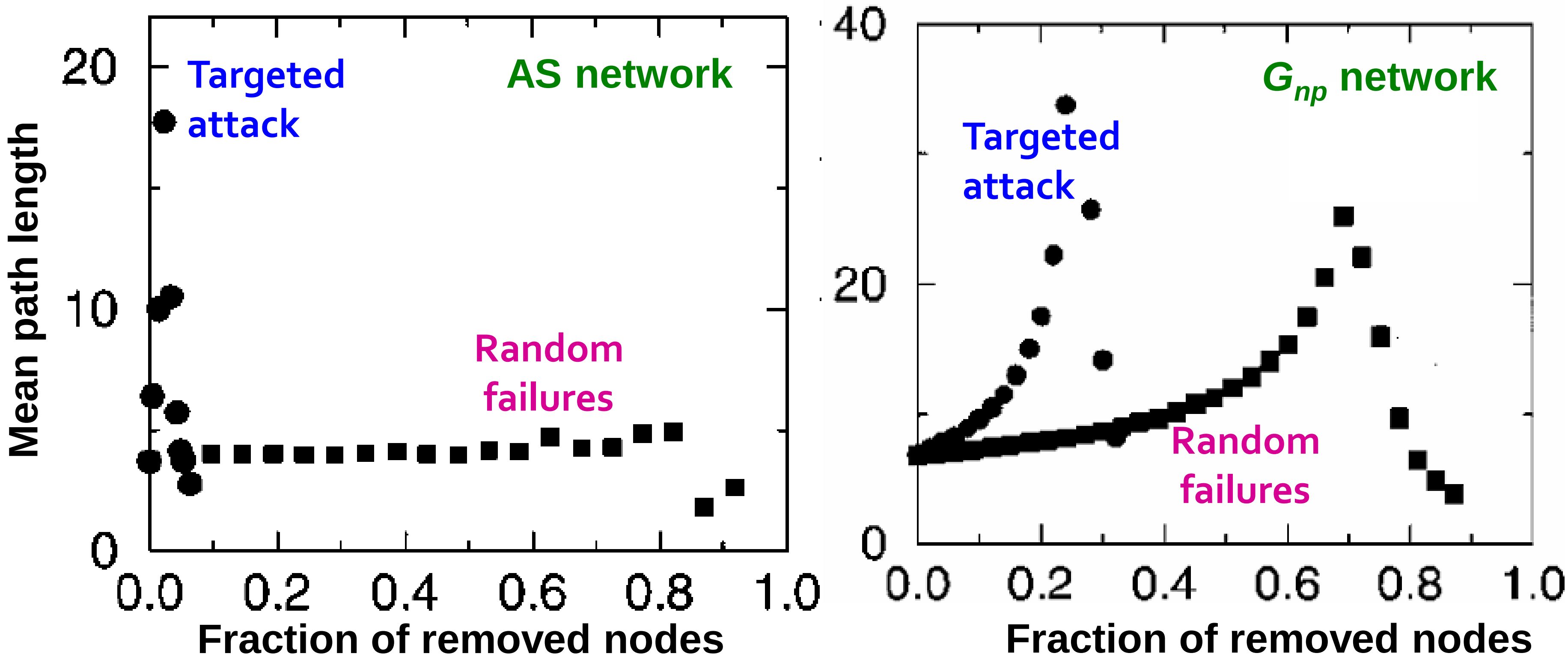
Degree distribution is Power-law

Consequence: Network Resilience

- **How does network connectivity change as nodes get removed?** [Albert et al. 00; Palmer et al. 01]
- **Nodes can be removed:**
  - **Random failure:**
    - Remove nodes uniformly at random
  - **Targeted attack:**
    - Remove nodes in order of decreasing degree
- This is important for **robustness of the internet** as well as **epidemiology**



Network Resilience



- **Real networks are resilient to random failures**
- **G_{np} has better resilience to targeted attacks**
  - Need to remove all pages of degree **>5** to disconnect the Web
  - But this is a very small fraction of all web pages