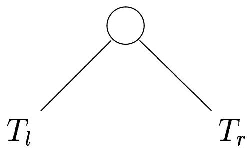
We only consider nonempty trees. Furthermore, we restrict ourselves to binary trees the nodes of which have either two or no children.

PROPOSITION 6.11 *In a binary tree T, the number of external nodes is 1 more than the number of internal nodes.*

PROOF We prove this proposition by structural induction on T. The base case, where T consists of a single node, is trivial. Next we consider the induction step. The binary tree T is of the form



where the left subtree T_l and the right subtree T_r are both smaller than T. By the induction hypothesis, the number of external nodes of T_l and T_r (denoted by e_l and e_r) are both 1 more than the number of internal nodes of T_l and T_r (denoted by i_l and i_r):

$$\begin{aligned} e_l &= i_l + 1 \\ e_r &= i_r + 1. \end{aligned}$$

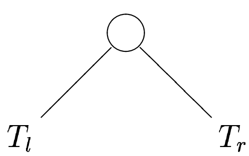
The number of external nodes of T is $e_l + e_r$ and the number of internal nodes of T is $i_l + i_r + 1$. We have that

$$\begin{aligned} e_l + e_r &= i_l + 1 + i_r + 1 \\ &= (i_l + i_r + 1) + 1. \end{aligned}$$

□

PROPOSITION 6.10.1 *Let T be a binary tree with e external nodes and height h. Then $h + 1 \leq e \leq 2^h$.*

PROOF We prove this claim by structural induction on T. The base case, where T consists of a single node, is trivial ($h = 0$ and $e = 1$). Next we consider the induction step. The binary tree T is of the form



Since the subtrees T_l and T_r are smaller than T, by the induction hypothesis the claim holds for these subtrees. Assume that the subtree T_l (T_r) has e_l (e_r) external nodes and height h_l (h_r). Then

$$\begin{aligned} h + 1 &\leq h_l + 1 + h_r + 1 \quad [h = \max\{h_l, h_r\} + 1] \\ &\leq e_l + e_r \quad [\text{induction hypothesis}] \\ &= e \\ &= e_l + e_r \\ &\leq 2^{h_l} + 2^{h_r} \quad [\text{induction hypothesis}] \\ &\leq 2 \cdot 2^{\max\{h_l, h_r\}} \\ &= 2^h \quad [h = \max\{h_l, h_r\} + 1] \end{aligned}$$

$\square$

PROPOSITION 6.10.2 *Let T be a binary tree with i internal nodes and height h. Then $h \leq i \leq 2^h - 1$.*

PROOF Let e be the number of external nodes of T. According to Proposition 6.11, $i = e - 1$. According to Proposition 6.10.1, $h + 1 \leq e \leq 2^h$. Combining these two results, we immediately obtain $h \leq i \leq 2^h - 1$. $\square$

PROPOSITION 6.10.3 *Let T be a binary tree with n nodes and height h. Then $2h + 1 \leq n \leq 2^{h+1} - 1$.*

PROOF Let e and i be the number of internal and external nodes of T. Clearly, $n = e + i$. Adding Proposition 6.10.1 and 6.10.2, we arrive at $2h + 1 \leq n \leq 2^{h+1} - 1$. $\square$

PROPOSITION 6.10.4 *Let T be a binary tree with n nodes and height h. Then $\log(n+1) - 1 \leq h \leq \frac{n-1}{2}$.*

PROOF According to Proposition 6.10.3,

$$
\begin{aligned}
& n \leq 2^{h+1} - 1 \\
& \quad \Leftrightarrow \quad n + 1 \leq 2^{h+1} \\
& \quad \Leftrightarrow \quad \log(n+1) \leq h + 1 \\
& \quad \Leftrightarrow \quad \log(n+1) - 1 \leq h
\end{aligned}
$$

and

$$
\begin{aligned}
& 2h + 1 \leq n \\
& \quad \Leftrightarrow \quad 2h \leq n - 1 \\
& \quad \Leftrightarrow \quad h \leq \tfrac{n-1}{2}.
\end{aligned}
$$

$\square$