PROPOSITION 9.11. *Let $G = (V, E)$ be a simple undirected nonempty graph with n vertices and m edges.*

1. *If G is connected then $m \geq n - 1$.*
2. *If G has no cycles then $m \leq n - 1$.*
3. *If G is connected and has no cycles then $m = n - 1$.*

PROOF

1. We prove this property by induction on n. The base case ($n = 1$, and hence $m = 0$) is vacuously true. Next we consider the induction step. Let $n > 1$. Pick a vertex $v \in V$. Let the degree of v be d. Let G' be the subgraph of G obtained by removing vertex v and all edges incident on v from G. Graph G' consists of at most d connected components. To each connected component we can apply the induction hypothesis. Hence, the number of edges of $G' \geq$ number of nodes of $G' - d$. Consequently,

$$\begin{aligned} m &= \text{number of edges of } G \\ &= \text{number of edges of } G' + d \\ &\geq \text{number of nodes of } G' \\ &= \text{number of nodes of } G - 1 \\ &= n - 1. \end{aligned}$$

2. Left as an exercise.
3. Immediate consequence of 1. and 2.

$\square$