Fault-Tolerate Gathering Algorithms for Autonomous Mobile Robots

N. AGMON AND D. PELEG
Outline

- Why gathering algorithms
- Definition of gathering and fault-tolerance in robotic
- Model parameters
- Impossibility of gathering under (3,1) — Byzantine failures in asynchronous and semi-synchronous systems
- An algorithm for Fully-synchronous gathering with $N \geq 3f + 1$
- The algorithm in practice
Why Cooperative Activities by Robots?

- Perform tasks not possible by a single robot.
- Decreases the cost of operation.
- Various applications
  - Military
  - Space mission e.g. exploration
Types of pattern formation problems

- Gathering and convergence
- Flocking (following a leader)
- Even Distribution
- Partitioning
The Gathering Problem

- N autonomous robots to occupy a single point within a finite number of steps.
- Similar to *Convergence* problem in message passing systems.
Fault-Tolerance

- $N$ robots with $f$ faulty ones should reach a point.
- Only simple failures are addressed before:
  - **Transient failure** (robots get lost)
  - **Sensor failure** which is known to other robots
- This paper addresses crash and **Byzantine** failures
Robot operation cycle

- Every Cycle include:
  - **Look**: identify the locations of all robots (ids *Unknown*)
  - **Compute**: execute an algorithm to choose a goal point $P_G$
  - **Move**: move towards $P_G$, at least by a distance $S$
Synchronization models

1. **Semi-Synchronous (SSYNC):** robots use same clock but not necessarily active in all cycles.

2. **Fully-Asynchronous (ASYNC):** every robot acts independently.

3. **Fully-Synchronous (FSYNC):**
   - All robots are active in all cycles
   - A lower and upper bound for maximum movements
Proposed model assumption

- Robots are assumed *Oblivious*
  - In dynamic environment knowledge is not useful
  - Obliviousness is the *worst case* scenario

- The only input is the set of positions $P$ of all robots.

- Robots are *Transparent*
The role of “Adversary”

- An external adversary is assigned that can,
  - Decide the distance a non-faulty robot can travel (no less than $S$)
  - Define arbitrary action for faulty robots
Gathering Under Byzantine Failure

- It is impossible to perform gathering in SSYNC and ASYNC models
  - If a problem is solvable in ASYNC, it is also in SSYNC
  - If prove no solvable in SSYNC, it also proves not solvable in ASYNC
Definition 1

- A gathering algorithm is *hyperactive* if it instructs every robot to make a move in every cycle.
Theorem 1

- $N = 3$ and $f = 1$, under the SSYNC model $\Rightarrow$ no non-hyperactive algorithm exists for gathering or convergence
Theorem 1 (proof)

- In $C_1$, $R_1$ active and instructed to stay and $R_2$ is passive i.e. doesn’t move
- Since $R_2$ in $C_2$ is in the same state as $R_1$ in $C_1$, it stays
- Adversary can switch from $C_1$ to $C_2$, forcing $R_1$ and $R_2$ to stay in place indefinitely.
Definition 2

- An algorithm is \( N - diverging \) if the distance of two non-faulty robots increases after a cycle.
Lemma 1

- In the SSYNC (or even FSYNC) model a 3 – *diverging* algorithm will fail to achieve gathering or convergence.
Lemma 1(proof)

- Suppose \( \exists \) an algorithm \( A \)

- Consider a \((3,1)\) – Byzantine system \( T \) with robots \( R_1, R_2 \) and \( R_3 \)

- \( \sigma = \{ C_0, C_1, ..., C_k \} \) is a sequence of configurations

- Adversary only intervenes in \( C_0 \) to \( C_1 \) and increases \( \text{dist}(R_1, R_2) \), i.e. \( d_1 > d_0 \)

- At \( C_k \) all robots are gathered in one point
Lemma 1(proof)

- Assume a system $T'$

\[ p_3(t'-1) \quad p_2(t') \quad p_2^m \quad z_2 \quad p_2(t'-1) \]

\[ p_1(t') \quad d_{t'} \quad d_0 \quad d_{t'-1} \quad C_{t'-1} \]

\[ p_1(t'-1) \quad p_1^m \quad z_1 \]
Lemma 1(proof)

- $R_3$ is faulty and $R_1$ and $R_2$ are stopped at $p_1$ and $p_2$. 

\[ C_{t'} = C_0 \]

\[ p_3(0) \quad p_2(0) \]

\[ d_0 \]

\[ p_1(0) \]

\[ p_3(t' - 1) \quad p_2(t' - 1) \]

\[ p_2^m \quad p_1^m \]

\[ d_0 \quad d_{t' - 1} \]

\[ C_{t' - 1} \]
Observation

- Let $A$ be an algorithm operating in $(3,1) - Byzantine$ system

- In any of the following scenarios, $A$ will be 3-diverging:
  - $C1$ - $0 \leq \mu_i \leq \pi \leq \mu_j \leq 2\pi$ OR $0 \leq \mu_j \leq \pi \leq \mu_i \leq 2\pi$
  - $C2$ - $0 \leq \mu_i < \mu_j \leq \pi$ AND $\mu_i \geq \frac{\pi}{2}$ OR $\mu_j \leq \frac{\pi}{2}$
  - $C3$ - $0 \leq \mu_j \leq \mu_i \leq \pi$
  - $C4$ - $\pi \leq \mu_i \leq \mu_j \leq 2\pi$
  - $C5$ - $\pi \leq \mu_j < \mu_i \leq 2\pi$ and either $\mu_i \leq \frac{3\pi}{2}$ or $\mu_j \leq \frac{3\pi}{2}$
Theorem 2

- In a $(3, 1)$ – Byzantine system under the SSYNC model it is impossible to perform successful gathering or convergence.
Theorem 2 (proof)

- Algorithm $A$ in which $R_1$, $R_2$ and $R_3$ are collinear, and $R_2$ is in middle.
- If $R_2 = \text{stationary} \implies \text{non-hyperactive} \implies$ by Theorem 1 gathering is not possible.

![Diagram showing the positions of $R_1$, $R_2$, and $R_3$ with angles $\mu_1$ and $\mu_3$. The notation $p_1$, $p_2$, and $p_3$ indicate the positions, with $p_2$ in the middle, and the notation "No Move" indicates the non-movement scenario.]
Theorem 2 (proof)

- From Observation \( \Rightarrow \) if \( 0 \leq \mu_1, \mu_2, \mu_3 \leq \pi \), to avoid 3-diverging, necessarily \( \mu_3 > \mu_2 > \mu_1 \).

  a) If \( \mu_2 \geq \frac{\pi}{2} \) \( \Rightarrow \) by C2, \( p_2 \) and \( p_3 \) are diverging.

  b) If \( \mu_2 \leq \frac{\pi}{2} \) \( \Rightarrow \) by C2, \( p_1 \) and \( p_2 \) are diverging.

- Similar argument for \( \pi \leq \mu_1, \mu_2, \mu_3 \leq 2\pi \).
Theorem 2 (proof)

- By $C_1 \Rightarrow$ If $\mu_1 > \pi$ and $\mu_2, \mu_3 < \pi$ OR If $\mu_1, \mu_2 > \pi$ and $\mu_3 < \pi$ A is diverging

- By Lemma 1, A fails to achieve gathering or convergence.
Fault tolerant gathering in the FSYNC model

- **Geometric span** of the set of point P:

  $$\text{Span}(P) = \max\{\text{dist}(p, q) | p, q \in P\}$$

- **The center of gravity** of a multiset P of $n \geq 3$ points $p_i = (x_i, y_i)$:

  $$C_{\text{grav}}(P) = \left( \frac{\sum_{i=1}^{N} x_i}{N}, \frac{\sum_{i=1}^{N} y_i}{N} \right)$$
Definitions

- A distributed robot algorithm is **Concentrating** if,
  1. It is non-diverging
  2. Exist a constant \( c > 0 \) at least one pair of non-faulty robots get closer by \( c \).

- The **Hull Intersection** \( H_{\text{int}}^k(P) \) is the convex set created as the intersection of all \( \binom{N}{k} \) sets \( H(P\setminus\{p_{i1}, \ldots, p_{ik}\}) \), for \( 1 \leq k \leq N, p_{ij} \in P \).
A gathering algorithm for $N \geq 3f + 1$ in the FSYNC model

The Algorithm

Procedure $Gather_{Byz}(P)$

1. Compute $Q \leftarrow V_H(H_{int}^f(P))$.
2. Set $p_G \leftarrow C_{grav}(Q)$.

$V_H$ denotes the set of vertices of $H_{int}^f(P)$.
Analysis

- The Objective is to show if,
- K Robots at points $P = \{p_1, \ldots, p_K\}$ move towards a point $p_G$ in their convex hull $H(P)$
- Their geometric span decreases by at least $cS$ for some constant $c \geq 1/4$
- The robots meet within finite states
Some Lemmas

**Lemma 2**: Two robots $R_1$ and $R_2$, and let $\alpha = \angle p_1p_Gp_2$.

If $\alpha \leq \frac{\pi}{2}$ then the distance between them decreases by at least $S'(1 - \cos \alpha)$.

\[d_1^2 = (a + S')^2 + (b + S')^2 - 2(a + S')(b + S')\cos \alpha\]
\[d_2^2 = a^2 + b^2 - 2abc\cos \alpha\]
\[d_1^2 - d_2^2 = (d_1 - d_2)(d_1 + d_2) \Rightarrow d_1 - d_2 = \frac{2a + 2b + 2S'}{d_1 + d_2} \cdot S'(1 - \cos \alpha)\]

$\Delta p_1p_Gp_2 \Rightarrow a + b + 2S' > d_1$, $\Delta p_1'p_Gp_2' \Rightarrow a + b > d_2$

\[\frac{2a + 2b + 2S'}{d_1 + d_2} > 1 \Rightarrow d_1 - d_2 > S'(1 - \cos \alpha)\]
Some Lemmas

- **Lemma 3**: $\alpha \geq \frac{\pi}{2} \Rightarrow \text{dist}(p_1, p_2)$ decreases by at least 0.7S

- **Proof**: given $d_2 \leq d_3 \Rightarrow \text{suffice to show } \Delta = d_1 - d_3 > 0.7S$

- $\Delta' = \text{dist}(p_0, p_1) \leq \Delta \text{ show } \Delta' \geq 0.7S$

- $\alpha \geq \frac{\pi}{2} \Rightarrow \beta + \gamma \leq \frac{\pi}{2}$. w.l.o.g. assume $\beta \leq \frac{\pi}{4}$

- Given $S_i \geq S$, by **sine theorem** on triangle $\Delta p_1 p_1' p_0$,

  \[S \leq S_1 = \frac{S_1}{\sin \left(\frac{\pi}{2}\right)} = \frac{\Delta'}{\sin \left(\frac{\pi}{2} - \beta\right)}\]

- Hence, $\Delta' \geq S. \cos(\beta) \geq S. \cos \left(\frac{\pi}{4}\right) \geq 0.7S$
Some Lemmas

**Lemma 4:** \( \text{Span}(P) = \text{dist}(p_a, p_b) \Rightarrow \) for every point \( P_G \) in \( H(P) \), \( \angle p_a p_G p_b \geq \pi/4 \).

**Proof:**

• By contradiction assume \( \alpha < \frac{\pi}{4} \) and w.l.o.g \( \beta \geq \gamma \)

\[ \alpha < \frac{\pi}{4} < \frac{3\pi}{8} < \frac{\pi - \alpha}{2} = \frac{\beta + \gamma}{2} < \beta + \gamma = \pi - \alpha \Rightarrow \sin \beta > \sin \alpha \]

• On \( \triangle p_a p_b p_G \), \( \frac{\text{dist}(p_a, p_b)}{\text{dist}(p_a, p_G)} = \frac{\sin \alpha}{\sin \beta} \)

• \( \text{dist} (p_a, p_G) > \text{dist} (p_a, p_b) = \text{Span} (P) \Rightarrow \) contradiction
Lemma 5

- $K$ robots $R_1, ..., R_K$ at $P = \{p_1, ..., p_K\}$
- Traverse same distance $S$ towards a point $p_G$ in the convex hull $H(P)$,
- New positions are $P' = \{p'_1, ..., p'_K\} \Rightarrow$
- $\text{Span}(P') \leq \text{Span}(P) - cS$, $c \geq 1/4$
Lemma 5 (proof)

- $p_a, p_b \in V_H(H(P))$ and $\text{Span}(P) = \text{dist}(p_a, p_b)$
- $p_a', p_b' \in V_H(H(P'))$ and $\text{Span}(P') = \text{dist}(p_a', p_b')$

- By Lemma 4 $\alpha = \angle p_a p_G p_b \geq \pi/4$.

- If $\frac{\pi}{4} \leq \alpha < \frac{\pi}{2}$, according to Lemma 2,
  \[ \text{dist}(p_a', p_b') \leq \text{dist}(p_a, p_b) - (1 - \cos \alpha) S \leq \text{dist}(p_a, p_b) - 0.25S. \]

- If $\alpha \geq \pi/2$, by Lemma 3
  \[ \text{dist}(p_a', p_b') \leq \text{dist}(p_a, p_b) - 0.7S \]

- So in any case we have,
  \[ \text{dist}(p_a', p_b') \leq \text{dist}(p_a, p_b) - 0.25S \Rightarrow \text{Span}(P') \leq \text{Span}(P) - 0.25S \]
Corollary 1

- From Lemma 5 we conclude,
- If a set of $K$ robots traverse at least by $S$,
- $\text{Span } (P') \leq \text{Span}(P) - cS$. 
Lemma 6

- Using the algorithm, Robots meet finite number of cycles
- **Proof:** for $t \geq 1$, $H_t$ = convex hull at the beginning of cycle $t$.
- Robots move at least $S$ in each cycle towards $p_G$ in the convex hull
- By **Corollary 1** $\Rightarrow Span(H_{t+1}) \leq Span(H_t) - 0.25S$ for every $t$
- At most $4. Span(H_1)/S$ cycles $\Rightarrow Span(P) = 0$.
- Thus all robots meet.
Theorem 3

- Algorithm $\text{Gather}_{byz}$ solves $(N, f) - Byzantine$ gathering for any $N \geq 3f + 1$, in $\text{FSYNC}$
- **Proof:** by Lemma 6 it is sufficient to show that $p_G \in H(R_{NF})$
- Set $H_{int}^f(P) \subseteq H(P)$ as well as every $N - f$ subsets of $P \Rightarrow H_{int}^f(P) \subseteq H(R_{NF})$
- $C_{grav}(P) \in H(P) \Rightarrow p_G \in H(R_{NF})$
- Therefore, $C_{grav}$ of the set $V_H\left( H_{int}^f(P) \right)$ is well defined.

*This concludes the proof of the algorithm*
The algorithm in practice
Timing of the algorithm

![Graph showing the timing of the algorithm with the x-axis representing the number of robots (Faulty) and the y-axis representing seconds. The graph indicates a sharp increase in seconds as the number of faulty robots increases.]
Remarks

- A Formal analysis of Gathering problem
- Fault tolerance for crash and Byzantine
- Offers solutions for $N \geq 3f$ crash and $N \geq 3f + 1$ Byzantine failure models
- In practice, only viable for few number of failures
- No speculation on upper bound for move
Helly’s theorem

Helly’s Theorem for $d = 2$ (cf. [27, Theorem 4.1.1]): Let $\mathcal{A}$ be a finite family of at least three convex sets in $\mathbb{R}^2$. If every three members of $\mathcal{A}$ have a point in common, then there is a point common to all members of $\mathcal{A}$.

**Lemma 6.1.** For a multiset $P = \{p_1, \ldots, p_N\}$, $N \geq 3k+1$, $H_{\text{int}}^k(P)$ is convex and nonempty.

**Proof:** $H_{\text{int}}^k(P)$ is convex as it is the intersection of $\binom{N}{k}$ convex sets. We prove that it is nonempty by Helly’s Theorem. Consider three arbitrary sets $P_l = \{p_{l1}, \ldots, p_{lk}\} \subseteq P$, $1 \leq l \leq 3$, and let $Q_l = H(P \setminus P_l)$, $1 \leq l \leq 3$. Then $Q_l \cap Q_j \cap Q_l$ contains at least $P' = P \setminus (P_1 \cup P_2 \cup P_3)$. As $|P| \geq 3k + 1$, $|P'| \geq 1$. It follows that the intersection of every three such sets is nonempty, and by Helly’s Theorem $V(H_{\text{int}}^k(P))$ is nonempty as well. ■