De Bakker-Zucker Processes Revisited

Franck van Breugel

Technical Report CS-1999-05

November 9, 1999

Department of Computer Science

4700 Keele Street, Toronto, Ontario M3J 1P3, Canada

De Bakker-Zucker Processes Revisited

Franck van Breugel*

York University, Department of Computer Science
4700 Keele Street, Toronto, Canada M3J 1P3

franck@cs.yorku.ca

Dedicated to Jaco de Bakker on the occasion of his 60th birthday

Abstract

The sets of compact and of closed subsets of a metric space endowed with the Hausdorff metric are studied. Both give rise to a functor on the category of 1-bounded metric spaces and nonexpansive functions. It is shown that the former functor has a terminal coalgebra and that the latter does not.

Introduction

In the seventies, the use of trees was quite popular in denotational semantics of programming languages. Infinite computations were modelled by infinite trees. These infinite trees were obtained by providing the set of finite trees with an order and by completing the ordered space. In the late seventies, Maurice Nivat defined a distance function on finite trees. When this distance function turned out to be a metric, even an ultrametric, the following question arose naturally: How are the completed ordered space of finite trees and the completed metric space of finite trees related? It turned out that the latter is the set of maximal elements of the former. This result may be seen as the start of the use of metric spaces in denotational semantics. It was published by André Arnold and Maurice Nivat in [AN80]. At the Third Advanced Course on Foundations of Computer Science, held at the Mathematical Centre in Amsterdam in August/September 1978, Nivat presented joint work with Arnold about metric spaces and how they can be used to give semantics to recursive program schemes. Nivat's lecture notes [Niv79] are his most cited publication.

Together with Jan van Leeuwen, Jaco de Bakker organized this course. At that time, De Bakker was completing his book [Bak80]. Jeff Zucker contributed an appendix to the book and assisted in preparing the final version. In the summer of 1981, De Bakker visited Zucker at Bar-Ilan University. Inspired by Nivat's work, they addressed the following question: Can metric spaces be used in denotational semantics of concurrency? Several visits of Zucker to Amsterdam followed and led to various publications including [BZ82]. In the latter paper, metric spaces were successfully exploited to give denotational semantics to various languages with concurrency. Besides showing that metric spaces can be used for that purpose, De Bakker and Zucker also demonstrated in [BZ82] how to solve recursive equations over metric spaces. Amongst others, they solved the equation

$$X = \mathcal{P}_c\left(A \times \tfrac{1}{2} \cdot X\right),$$

where A is a set endowed with discrete metric, $\frac{1}{2}\cdot$ multiplies the metric of a metric space by a half, and $\mathcal{P}_c$ denotes the set of closed subsets of a metric space endowed with the Hausdorff metric. That is, they constructed a metric space X which is isometric to $\mathcal{P}_c\left(A \times \frac{1}{2} \cdot X\right)$. In [BZ83], the closely related equation

$$X = \mathcal{P}_k\left(A \times \tfrac{1}{2} \cdot X\right),$$

where $\mathcal{P}_k$ denotes the set of compact subsets of a metric space endowed with the Hausdorff metric, was solved. The elements of these two metric spaces are sometimes called De Bakker-Zucker processes. These

*Supported by Natural Sciences and Engineering Research Council of Canada.

processes can be viewed as tree-like structures. They have been used as denotations of programs of a large variety of languages. The metric structure on these processes admits the modelling of infinite computations as limits of finite computations.

[BZ82] is the first paper of the Amsterdam Concurrency Group. This group was (and still is) directed by De Bakker and consists of members of CWI's department of software technology and a number of close affiliates at Dutch universities. The main focus of this group has been on the use of metric spaces to give semantics to programming languages and to relate different semantic models for a given language. A selection of papers by members of the Amsterdam Concurrency Group can be found in [BR92]. Two of its members, Jaco de Bakker and Erik de Vink, wrote a textbook on metric semantics [BV96].

Bisimulation has been a key notion in concurrency theory for the last two decades. This notion is due to Robin Milner and David Park [Mil80, Par81, Mil94]. Not long after the introduction of De Bakker-Zucker processes, several members of the Amsterdam Concurrency Group suspected that these processes are closely related to bisimulation (see, e.g., [BBKM84]). More than half a decade later, Rob van Glabbeek and Jan Rutten made this suspicion precise. In [GR89], they showed that De Bakker-Zucker processes represent bisimulation equivalence classes.

In the early eighties, a link between recursive equations over ordered spaces and terminal coalgebras was established. During a train ride between Amsterdam and Eindhoven in May 1989, Gordon Plotkin sketched to Jan Rutten how recursive equations over metric spaces can be solved exploiting the techniques used in the order-theoretic setting. A few years later, Rutten worked out all the details [RT92]. Rutten also showed that the obtained solutions are carriers of terminal coalgebras. Furthermore, he demonstrated that the above introduced equations defining the metric spaces of De Bakker-Zucker processes can be solved in this way and hence are carriers of terminal coalgebras. Terminal coalgebras provide us with coinductive definitions on the elements of the solution of the equations and coinductive proofs, a powerful technique for proving properties of those elements (see, e.g., [JR97] for more details).

In the original equations of [BZ82, BZ83], the multiplication of the metric by a half—denoted above by $\frac{1}{2}\cdot$ —was not made explicit (cf. [BR92, page 79]). Only in the constructions of the solutions of the equations the halves turned up. These halves were overlooked by Javier Thayer. In [Tha87], he sketched that there does not exist an ultrametric space which is the solution of the equation

$$X = \mathcal{P}_c(A \times X) \tag{1}$$

(for the details, see [BW99]). In the construction of the solution of a closely related equation, De Bakker and Zucker forgot one half. This missing half was spotted by Mila Majster-Cederbaum and Frank Zetzsche [MZ91]. They showed that the space constructed by De Bakker and Zucker (with the missing half) is not a solution of the equation. The presence of the halves (or some other positive constant smaller than 1) is essential for De Bakker and Zucker's method for solving recursive equations and for the generalization of their method by Pierre America and Jan Rutten [AR89]. They are also a key ingredient of Rutten's above mentioned metric terminal coalgebra theorem.

In this paper, we consider the following questions: What happens if we leave out the halves? Do we still have terminal coalgebras? We consider the equations

$$X = \mathcal{P}_c(X) \text{ and } X = \mathcal{P}_k(X).$$

We show that there does not exist a terminal coalgebra which solves the first equation and that there exists a terminal coalgebra which solves the second one. The former result is based on Cantor's theorem. The proof of the latter contains the following three main ingredients: an adjunction from the category of 1-bounded metric spaces to the category of sets, a result of Claudio Hermida and Bart Jacobs about adjunctions between categories of coalgebras, and a result of Michael Barr about the existence of terminal coalgebras in the category of sets. These results can easily be extended to equations like (1),

$$X = \mathcal{P}_k(A \times X) \text{ and } X = A \to \mathcal{P}_k(X).$$

Of course, we still have to answer the following question: Why would we drop the halves? First of all, by dropping the halves we obtain simpler equations. If we can solve these equations, as we can in the compact case, we may want to use them instead of the equations with the halves. We cannot solve equation (1). However, the closed case (even with the half) is known to be problematic (see, e.g., [BR92, page 130]).

Secondly, in some cases the metric structure can be used other than for modelling infinite computations as limits. It may be exploited to capture the difference of computations quantitatively. For example, metrics have been used in this way for probabilistic systems (see, e.g., [GJS90, DGJP99]). In those cases, we don't want any halves in the equations as they blur the quantitative picture.

We assume that the reader is familiar with metric spaces, coalgebras and categories. For more details we refer the reader to, e.g., [BV96, JR97, Mac71].

Acknowledgements

The author has benefited from discussions with Jaco de Bakker, Michael Barr, Claudio Hermida, Maurice Nivat, Jan Rutten, Stephen Watson and Jeff Zucker.

1 Quotient of a metric space

Given a 1-bounded metric space, we will define an equivalence relation on the underlying set. The set of equivalence classes "best approximates" the metric space as we will see in the next section. Furthermore, we will prove a few properties of the equivalence relation, which we will exploit in later sections.

The equivalence relation is introduced in

DEFINITION 1 Let X be a metric space. Let $\sim$ be the smallest equivalence relation on (the set underlying) X containing

$$\stackrel{1}{\sim} = \{ \langle x, y \rangle \in X \times X \mid d(x, y) < 1 \}.$$

┘

PROPOSITION 2 *Let X and Y be metric spaces. Let $f : X \to Y$ be a nonexpansive function. If $x \sim y$ then $f(x) \sim f(y)$.*

PROOF It suffices to prove that $x \stackrel{1}{\sim} y$ implies $f(x) \stackrel{1}{\sim} f(y)$. This follows immediately from the nonexpansiveness of f. □

PROPOSITION 3 *Let X be a metric space. Let Y be endowed with the discrete metric. Let $f : X \to Y$ be a nonexpansive function. If $x \sim y$ then $f(x) = f(y)$.*

PROOF Similar to the proof of Proposition 2 exploiting the discreteness of Y. □

Given a 1-bounded metric space X, the set $\mathcal{P}_c(X)$ of closed subsets of X endowed with the Hausdorff metric is also a 1-bounded metric space. The equivalence relations of these two metric spaces are related as follows.

PROPOSITION 4 *Let X be a 1-bounded metric space. For all $A, B \in \mathcal{P}_c(X)$,*

if $A \sim B$ then $\forall a \in A : \exists b \in B : a \sim b$ and $\forall b \in B : \exists a \in A : a \sim b$.

PROOF It suffices to observe that

$$\begin{aligned}
& A \stackrel{1}{\sim} B \\
\Leftrightarrow\ & d_{\mathcal{P}_c(X)}(A, B) < 1 \\
\Leftrightarrow\ & \sup_{a \in A} \inf_{b \in B} d_X(a, b) < 1 \text{ and } \sup_{b \in B} \inf_{a \in A} d_X(a, b) < 1 \\
\Leftrightarrow\ & \forall a \in A : \inf_{b \in B} d_X(a, b) < 1 \text{ and } \forall b \in B : \inf_{a \in A} d_X(a, b) < 1 \\
\Leftrightarrow\ & \forall a \in A : \exists b \in B : d_X(a, b) < 1 \text{ and } \forall b \in B : \exists a \in A : d_X(a, b) < 1 \\
\Leftrightarrow\ & \forall a \in A : \exists b \in B : a \stackrel{1}{\sim} b \text{ and } \forall b \in B : \exists a \in A : a \stackrel{1}{\sim} b.
\end{aligned}$$

□

The implication in the other direction does not hold. Consider the set $\mathbb{N}$ endowed with the metric

$$d(m, n) = \min \{ \tfrac{1}{2} \cdot |m - n|, 1 \}.$$

Both $\{0\}$ and $\mathbb{N}$ are closed subsets of the metric space. Clearly, $\mathbb{N} \not\sim \{0\}$. However, $n \sim 0$ for all $n \in \mathbb{N}$.

2 An adjunction from metric spaces to sets

We present an adjunction from the category $\mathcal{M}et$ of 1-bounded metric spaces and nonexpansive functions to the category $\mathcal{S}et$. This adjunction will be exploited in the next section.

By endowing a set S with the discrete metric we obtain a 1-bounded metric space. This space is denoted by $\mathcal{D}(S)$. Each function $f : S \to T$ is a nonexpansive function from $\mathcal{D}(S)$ to $\mathcal{D}(T)$. Obviously, this defines a functor $\mathcal{D}$ from $\mathcal{S}et$ to $\mathcal{M}et$.

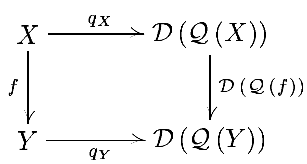
The functor $\mathcal{Q}$ assigns to each 1-bounded metric space X the set $\mathcal{Q}(X)$ of $\sim$-equivalence classes. The equivalence class containing $x \in X$ is denoted by $q_X(x)$. The functor $\mathcal{Q}$ maps a nonexpansive function $f : X \to Y$ to the function $\mathcal{Q}(f) : \mathcal{Q}(X) \to \mathcal{Q}(Y)$ defined by

$$\mathcal{Q}(f)(q_X(x)) = q_Y(f(x))$$

(cf. Proposition 2). Clearly, $\mathcal{Q}$ is a functor from $\mathcal{M}et$ to $\mathcal{S}et$.

Proposition 5 *$\mathcal{Q}$ is a left adjoint for $\mathcal{D}$.*

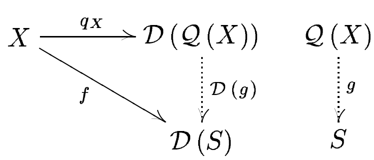
Proof Let X be a 1-bounded metric space. From the definition of $\sim$ and the fact that X is 1-bounded it follows that $q_X : X \to \mathcal{D}(\mathcal{Q}(X))$ is nonexpansive. For a 1-bounded metric space Y and a nonexpansive function $f : X \to Y$ we have that $q_Y \circ f = \mathcal{D}(\mathcal{Q}(f)) \circ q_X$, i.e. q is a natural transformation.



Hence, according to, e.g., [Mac71, Theorem IV.1.2(i)], it suffices to observe that for all 1-bounded metric spaces X and sets S and nonexpansive functions $f : X \to \mathcal{D}(S)$ there exists a unique function $g : \mathcal{Q}(X) \to S$ defined by

$$g(q_X(x)) = f(x)$$

(cf. Proposition 3) such that $\mathcal{D}(g) \circ q_X = f$.



□

3 Compact sets

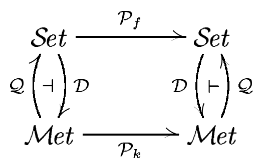
The functor $\mathcal{P}_k$ assigns to each 1-bounded metric space X the set of compact subsets of X endowed with the Hausdorff metric. A nonexpansive function $f : X \to Y$ is mapped to the nonexpansive function $\mathcal{P}_k(f) : \mathcal{P}_k(X) \to \mathcal{P}_k(Y)$ defined by

$$\mathcal{P}_k(f)(A) = \{ f(a) \mid a \in A \}.$$

This functor has a terminal coalgebra as is shown in

Proposition 6 *There exists a terminal $\mathcal{P}_k$-coalgebra.*

PROOF Let $\mathcal{P}_f$ denote the finite powerset functor on $\mathcal{S}et$. Since $\mathcal{D} \circ \mathcal{P}_f$ is isomorphic to $\mathcal{P}_k \circ \mathcal{D}$, we can conclude from Proposition 5 and [HJ98, Corollary 2.15] that there exists an adjunction from the category of $\mathcal{P}_k$-coalgebras to the category of $\mathcal{P}_f$-coalgebras.



According to [Bar93, Theorem 1.2], there exists a terminal $\mathcal{P}_f$-coalgebra. Since right adjoints preserve terminal objects (see, e.g., [Mac71, Theorem V.5.1]), there also exists a terminal $\mathcal{P}_k$-coalgebra. □

4 Closed sets

The functor $\mathcal{P}_c$ maps each 1-bounded metric space X to the set of closed subsets of X endowed with the Hausdorff metric. A nonexpansive function $f : X \to Y$ is assigned to the nonexpansive function $\mathcal{P}_c(f) : \mathcal{P}_c(X) \to \mathcal{P}_c(Y)$ defined by

$$\mathcal{P}_c(f)(A) = \text{the smallest closed set containing } \{ f(a) \mid a \in A \}.$$

This functor does not have a terminal coalgebra.

PROPOSITION 7 *There does not exist a terminal $\mathcal{P}_c$-coalgebra.*

PROOF Towards a contradiction, assume that $f : X \to \mathcal{P}_c(X)$ is a terminal $\mathcal{P}_c$-coalgebra. Hence, X is isomorphic to $\mathcal{P}_c(X)$ according to, e.g., [JR97, Lemma 6.4(ii)]. In the next paragraph, we will show that X carries the discrete metric. Since every subset of a discrete metric space is closed, we can conclude that X has the same cardinality as $\mathcal{P}(X)$. This contradicts Cantor's theorem.

We define the $\mathcal{P}_c$-coalgebra $g : \mathcal{D}(\mathcal{Q}(X)) \to \mathcal{P}_c(\mathcal{D}(\mathcal{Q}(X)))$ by

$$g(q_X(x)) = \{ q_X(y) \mid y \in f(x) \}.$$

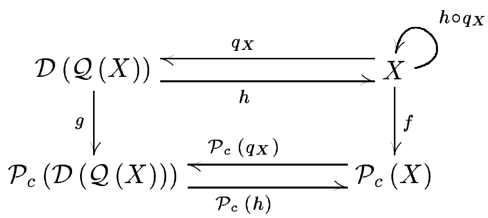
Note that for all $x_1, x_2 \in X$,

$$\begin{array}{ll} x_1 \sim x_2 & \\ \quad \Rightarrow \quad f(x_1) \sim f(x_2) & \text{[Proposition 2]} \\ \quad \Rightarrow \quad \forall y_1 \in f(x_1) : \exists y_2 \in f(x_2) : y_1 \sim y_2 \text{ and } \forall y_2 \in f(x_2) : \exists y_1 \in f(x_1) : y_1 \sim y_2 & \text{[Proposition 4]} \\ \quad \Rightarrow \quad \{ q_X(y_1) \mid y_1 \in f(x_1) \} = \{ q_X(y_2) \mid y_2 \in f(x_2) \}. & \end{array}$$

The function $q_X : X \to \mathcal{D}(\mathcal{Q}(X))$ is nonexpansive (cf. the proof of Proposition 5) and a $\mathcal{P}_c$-homomorphism since for all $x \in X$,

$$\begin{array}{ll} (g \circ q_X)(x) & \\ \quad = & g(q_X(x)) \\ \quad = & \{ q_X(y) \mid y \in f(x) \} \\ \quad = & \mathcal{P}_c(q_X)(f(x)) \\ \quad = & (\mathcal{P}_c(q_X) \circ f)(x). \end{array}$$

Let $h : \mathcal{D}(\mathcal{Q}(X)) \to X$ be the unique $\mathcal{P}_c$-homomorphism from the $\mathcal{P}_c$-coalgebra on $\mathcal{D}(\mathcal{Q}(X))$ to the terminal $\mathcal{P}_c$-coalgebra. Then $h \circ q_X$ is a $\mathcal{P}_c$-homomorphism from the terminal $\mathcal{P}_c$-coalgebra to itself. Also the identity function on X is such a $\mathcal{P}_c$-homomorphism. Again using the fact that the $\mathcal{P}_c$-coalgebra on X is terminal, we can conclude that $h \circ q_X$ is the identity function on X. Hence, q_X is one-to-one. Obviously, q_X is onto. Therefore, X and $\mathcal{D}(\mathcal{Q}(X))$ are isometric. Thus, X carries the discrete metric.



□

Note that there do exist metric spaces X which have the same cardinality as $\mathcal{P}_c(X)$. For example, the set $[0,1]$ endowed with the Euclidean metric and $\mathcal{P}_c([0,1])$ have the same cardinality (cf., e.g., [Jec97, page 32]).

References

[AN80] A. Arnold and M. Nivat. The Metric Space of Infinite Trees, Algebraic and Topological Properties. *Fundamenta Informaticae*, 3(4):445–476, 1980.

[AR89] P. America and J.J.M.M. Rutten. Solving Reflexive Domain Equations in a Category of Complete Metric Spaces. *Journal of Computer and System Sciences*, 39(3):343–375, December 1989.

[Bak80] J.W. Bakker. *Mathematical Theory of Program Correctness.* Series in Computer Science. Prentice Hall International, London, 1980.

[Bar93] M. Barr. Terminal Coalgebras in Well-Founded Set Theory. *Theoretical Computer Science*, 114(2):299–315, June 1993.

[BBKM84] J.W. de Bakker, J.A. Bergstra, J.W. Klop, and J.-J.Ch. Meyer. Linear Time and Branching Time Semantics for Recursion with Merge. *Theoretical Computer Science*, 34(1/2):135–156, November 1984.

[BR92] J.W. de Bakker and J.J.M.M. Rutten, editors. *Ten Years of Concurrency Semantics, selected papers of the Amsterdam Concurrency Group.* World Scientific, Singapore, 1992.

[BV96] J.W. de Bakker and E.P. de Vink. *Control Flow Semantics.* Foundations of Computing Series. The MIT Press, Cambridge, 1996.

[BW99] F. van Breugel and S. Watson. A Note on Hyperspaces and Terminal Coalgebras. In B. Jacobs and J.J.M.M. Rutten, editors, *Proceedings of the 2nd Workshop on Coalgebraic Methods in Computer Science*, volume 19 of *Electronic Notes in Theoretical Computer Science*, Amsterdam, March 1999. Elsevier Science.

[BZ82] J.W. de Bakker and J.I. Zucker. Processes and the Denotational Semantics of Concurrency. *Information and Control*, 54(1/2):70–120, July/August 1982.

[BZ83] J.W. de Bakker and J.I. Zucker. Compactness in Semantics for Merge and Fair Merge. In E. Clarke and D. Kozen, editors, *Proceedings of 4th Workshop on Logics of Programs*, volume 164 of *Lecture Notes in Computer Science*, pages 18–33, Pittsburgh, June 1983. Springer-Verlag.

[DGJP99] J. Desharnais, V. Gupta, R. Jagadeesan, and P. Panangaden. Metrics for Labeled Markov Systems. In J.C.M. Baeten and S. Mauw, editors, *Proceedings of the 10th International Conference on Concurrency Theory*, volume 1664 of *Lecture Notes in Computer Science*, Eindhoven, August 1999. Springer-Verlag.

[GJS90] A. Giacalone, C.-C. Jou, and S.A. Smolka. Algebraic Reasoning for Probabilistic Concurrent Systems. In *Proceedings of the IFIP WG 2.2/2.3 Working Conference on Programming Concepts and Methods*, Sea of Gallilee, April 1990. North-Holland.

[GR89] R.J. van Glabbeek and J.J.M.M. Rutten. The Processes of De Bakker and Zucker represent Bisimulation Equivalence Classes. In *J.W. de Bakker, 25 jaar semantiek*, pages 243–246. CWI, Amsterdam, 1989.

[HJ98] C. Hermida and B. Jacobs. Structural Induction and Coinduction in a Fibrational Setting. *Information and Computation*, 145(2):107–152, September 1998.

[Jec97] T.J. Jech. *Set Theory*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, second edition, 1997.

[JR97] B. Jacobs and J.J.M.M. Rutten. A Tutorial on (Co)Algebras and (Co)Induction. *Bulletin of the EATCS*, 62:222–259, June 1997.

[Mac71] S. Mac Lane. *Categories for the Working Mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, 1971.

[Mil80] R. Milner. *A Calculus of Communicating Systems*, volume 92 of *Lecture Notes in Computer Science*. Springer-Verlag, Berlin, 1980.

[Mil94] R. Milner. David Michael Ritchie Park (1935–1990) in memoriam. *Theoretical Computer Science*, 133(2):187–200, October 1994.

[MZ91] M.E. Majster-Cederbaum and F. Zetzsche. Towards a Foundation for Semantics in Complete Metric Spaces. *Information and Computation*, 90(2):217–243, February 1991.

[Niv79] M. Nivat. Infinite Words, Infinite Trees, Infinite Computations. In J.W. de Bakker and J. van Leeuwen, editors, *Foundations of Computer Science III*, part 2: Languages, Logic, Semantics, volume 109 of *Mathematical Centre Tracts*, pages 3–52. Mathematical Centre, Amsterdam, 1979.

[Par81] D. Park. Concurrency and Automata on Infinite Sequences. In P. Deussen, editor, *Proceedings of 5th GI-Conference on Theoretical Computer Science*, volume 104 of *Lecture Notes in Computer Science*, pages 167–183, Karlsruhe, March 1981. Springer-Verlag.

[RT92] J.J.M.M. Rutten and D. Turi. On the Foundations of Final Semantics: non-standard sets, metric spaces, partial orders. In J.W. de Bakker, W.-P. de Roever, and G. Rozenberg, editors, *Proceedings of the REX Workshop on Semantics: Foundations and Applications*, volume 666 of *Lecture Notes in Computer Science*, pages 477–530, Beekbergen, June 1992. Springer-Verlag.

[Tha87] F.J. Thayer. Letter to J.W. de Bakker, April 1987.