EECS 4401/5326 Winter 2022
Week 5 — $\mathcal{ALC}$ Tableau Proofs Examples

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Exercise 1

Use the tableau method for $\mathcal{ALC}$ described in Baader and Sattler's paper to check whether the following concept is satisfiable/consistent. Show the steps and rules that are used. If the concept is satisfiable give the model(s) (satisfying interpretation(s)) obtained by the method.

$$((\forall R.(\exists R.A)) \sqcap (\exists R.B)) \sqcap (\forall R.((\forall R.\neg A) \sqcup (\forall R.\neg B)))$$

Solution

Let

$$\begin{aligned}
C_0 &= ((\forall R.(\exists R.A)) \sqcap (\exists R.B)) \sqcap (\forall R.((\forall R.\neg A) \sqcup (\forall R.\neg B)))\\
C_1 &= ((\forall R.(\exists R.A)) \sqcap (\exists R.B))\\
C_2 &= (\forall R.((\forall R.\neg A) \sqcup (\forall R.\neg B)))
\end{aligned}$$

Start with

$$\mathcal{S} = \{\{C_0(x_0)\}\}$$

or graphically

$$x_o \quad \bullet \quad \{C_0\}$$

Apply $\rightarrow_\sqcap$ to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0)\}\}$$

or graphically

$$x_o \quad \bullet \quad \{C_0, C_1, C_2\}$$

Apply $\rightarrow_{\sqcap}$ again to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0)\}\}$$

or graphically

$$x_o \quad \bullet \quad \{C_0, C_1, C_2, \forall R.(\exists R.A), \exists R.B\}$$

Apply $\rightarrow_{\exists}$ to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\ R(x_0, x_1), B(x_1)\}\}$$

or graphically

$$\begin{array}{lcl} x_o & \bullet & \{C_0, C_1, C_2, \forall R.(\exists R.A), \exists R.B\} \\ & | & \\ & |\ R & \\ & | & \\ x_1 & \bullet & \{B\} \end{array}$$

Apply $\rightarrow_{\forall}$ to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\ R(x_0, x_1), B(x_1), \exists R.A(x_1)\}\}$$

or graphically

$$\begin{array}{lcl} x_o & \bullet & \{C_0, C_1, C_2, \forall R.(\exists R.A), \exists R.B\} \\ & | & \\ & |\ R & \\ & | & \\ x_1 & \bullet & \{B, \exists R.A\} \end{array}$$

Apply $\rightarrow_{\forall}$ to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\ R(x_0, x_1), B(x_1), \exists R.A(x_1), ((\forall R.\neg A) \sqcup (\forall R.\neg B))(x_1)\}\}$$

or graphically

$$\begin{array}{ccl} x_o & \bullet & \{C_0, C_1, C_2, \forall R.(\exists R.A), \exists R.B\} \\ & \mid R & \\ x_1 & \bullet & \{B, \exists R.A, ((\forall R.\neg A) \sqcup (\forall R.\neg B))\} \end{array}$$

Apply $\rightarrow_{\exists}$ to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\ R(x_0, x_1), B(x_1), \exists R.A(x_1), ((\forall R.\neg A) \sqcup (\forall R.\neg B))(x_1), \\ R(x_1, x_2), A(x_2)\}\}$$

or graphically

$$\begin{array}{ccl} x_o & \bullet & \{C_0, C_1, C_2, \forall R.(\exists R.A), \exists R.B\} \\ & \mid R & \\ x_1 & \bullet & \{B, \exists R.A, ((\forall R.\neg A) \sqcup (\forall R.\neg B))\} \\ & \mid R & \\ x_2 & \bullet & \{A\} \end{array}$$

Apply $\rightarrow_{\sqcup}$ to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\ R(x_0, x_1), B(x_1), \exists R.A(x_1), ((\forall R.\neg A) \sqcup (\forall R.\neg B))(x_1), \forall R.\neg B(x_1), \\ R(x_1, x_2), A(x_2)\}, \\ \{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\ R(x_0, x_1), B(x_1), \exists R.A(x_1), ((\forall R.\neg A) \sqcup (\forall R.\neg B))(x_1), \forall R.\neg A(x_1), \\ R(x_1, x_2), A(x_2)\}\}$$

or graphically for the first ABox

$$\begin{array}{ll}
x_o \bullet & \{C_0, C_1, C_2, \forall R.(\exists R.A), \exists R.B\} \\
\quad | \; R & \\
x_1 \bullet & \{B, \exists R.A, ((\forall R.\neg A) \sqcup (\forall R.\neg B)), \forall R.\neg B\} \\
\quad | \; R & \\
x_2 \bullet & \{A\}
\end{array}$$

Apply $\rightarrow_\forall$ to get

$$\begin{gathered}
\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\
R(x_0, x_1), B(x_1), \exists R.A(x_1), ((\forall R.\neg A) \sqcup (\forall R.\neg B))(x_1), \forall R.\neg B(x_1), \\
R(x_1, x_2), A(x_2), \neg B(x_2)\}, \\
\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\
R(x_0, x_1), B(x_1), \exists R.A(x_1), ((\forall R.\neg A) \sqcup (\forall R.\neg B))(x_1), \forall R.\neg A(x_1), \\
R(x_1, x_2), A(x_2)\}\}
\end{gathered}$$

or graphically for the first ABox

$$\begin{array}{ll}
x_o \bullet & \{C_0, C_1, C_2, \forall R.(\exists R.A)), \exists R.B)\} \\
\quad | \; R & \\
x_1 \bullet & \{B, \exists R.A, ((\forall R.\neg A) \sqcup (\forall R.\neg B)), \forall R.\neg B\} \\
\quad | \; R & \\
x_2 \bullet & \{A, \neg B\}
\end{array}$$

No clash and no further rules apply to the first ABox so C_0 is satisfiable.

A model I is $\Delta^I = \{x_0, x_1, x_2\}$, $A^I = \{x_2\}$, $B^I = \{x_1\}$, and $R^I = \{\langle x_0, x_1\rangle, \langle x_1, x_2\rangle\}$.

We can continue and apply $\rightarrow_{\forall}$ to the second ABox to get

$$\begin{gathered}\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0),\\ R(x_0, x_1), B(x_1), \exists R.A(x_1), ((\forall R.\neg A) \sqcup (\forall R.\neg B))(x_1), \forall R.\neg B(x_1),\\ R(x_1, x_2), A(x_2), \neg B(x_2)\},\\ \{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0),\\ R(x_0, x_1), B(x_1), \exists R.A(x_1), ((\forall R.\neg A) \sqcup (\forall R.\neg B))(x_1), \forall R.\neg A(x_1),\\ R(x_1, x_2), A(x_2), \neg A(x_2)\}\}\end{gathered}$$

or graphically for the second ABox

x_o • $\{C_0, C_1, C_2, \forall R.(\exists R.A)), \exists R.B)\}$

| R

x_1 • $\{B, \exists R.A, ((\forall R.\neg A) \sqcup (\forall R.\neg B)), \forall R.\neg A\}$

| R

x_2 • $\{A, \neg A\}$

There is a clash so this ABox is unsatisfiable. No further rules apply so there are no further models.

Exercise 2

Use the tableau method for $\mathcal{ALC}$ described in Baader and Sattler's paper to check whether the following concept is satisfiable/consistent. Show the steps and rules that are used. If the concept is satisfiable give the model(s) (satisfying interpretation(s)) obtained by the method.

$$((\forall R.(\exists R.A)) \sqcap (\exists R.B)) \sqcap (\forall R.((\forall R.\neg A) \sqcap (\forall R.\neg B)))$$

Solution

Let

$$\begin{aligned}C_0 &= ((\forall R.(\exists R.A)) \sqcap (\exists R.B)) \sqcap (\forall R.((\forall R.\neg A) \sqcap (\forall R.\neg B)))\\ C_1 &= ((\forall R.(\exists R.A)) \sqcap (\exists R.B))\\ C_2 &= (\forall R.((\forall R.\neg A) \sqcap (\forall R.\neg B)))\end{aligned}$$

Start with

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or graphically

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Apply $\rightarrow_\sqcap$ to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0)\}\}$$

or graphically

$$x_o \quad \bullet \quad \{C_0, C_1, C_2\}$$

Apply $\rightarrow_\sqcap$ again to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0)\}\}$$

or graphically

$$x_o \quad \bullet \quad \{C_0, C_1, C_2, \forall R.(\exists R.A), \exists R.B\}$$

Apply $\rightarrow_\exists$ to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\ R(x_0, x_1), B(x_1)\}\}$$

or graphically

$$\begin{array}{lll} x_o & \bullet & \{C_0, C_1, C_2, \forall R.(\exists R.A), \exists R.B\} \\ & | & \\ & | \; R & \\ & | & \\ x_1 & \bullet & \{B\} \end{array}$$

Apply $\rightarrow_\forall$ to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\ R(x_0, x_1), B(x_1), \exists R.A(x_1)\}\}$$

or graphically

$$\begin{array}{lll} x_o & \bullet & \{C_0, C_1, C_2, \forall R.(\exists R.A), \exists R.B\} \\ & | & \\ & | \; R & \\ & | & \\ x_1 & \bullet & \{B, \exists R.A\} \end{array}$$

Apply $\rightarrow_{\forall}$ to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\ R(x_0, x_1), B(x_1), \exists R.A(x_1), ((\forall R.\neg A) \sqcap (\forall R.\neg B))(x_1)\}\}$$

or graphically

$$\begin{array}{ccl} x_o & \bullet & \{C_0, C_1, C_2, \forall R.(\exists R.A)), \exists R.B)\} \\ & | & \\ & |\ R & \\ & | & \\ x_1 & \bullet & \{B, \exists R.A, ((\forall R.\neg A) \sqcap (\forall R.\neg B))\} \end{array}$$

Apply $\rightarrow_{\exists}$ to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\ R(x_0, x_1), B(x_1), \exists R.A(x_1), ((\forall R.\neg A) \sqcap (\forall R.\neg B))(x_1), \\ R(x_1, x_2), A(x_2)\}\}$$

or graphically

$$\begin{array}{ccl} x_o & \bullet & \{C_0, C_1, C_2, \forall R.(\exists R.A)), \exists R.B)\} \\ & | & \\ & |\ R & \\ & | & \\ x_1 & \bullet & \{B, \exists R.A, ((\forall R.\neg A) \sqcap (\forall R.\neg B))\} \\ & | & \\ & |\ R & \\ & | & \\ x_2 & \bullet & \{A\} \end{array}$$

Apply $\rightarrow_{\sqcap}$ again to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\ R(x_0, x_1), B(x_1), \exists R.A(x_1), \\ ((\forall R.\neg A) \sqcap (\forall R.\neg B))(x_1), \forall R.\neg A(x_1), \forall R.\neg B(x_1), \\ R(x_1, x_2), A(x_2)\}\}$$

or graphically

$$
\begin{array}{lcl}
x_o & \bullet & \{C_0, C_1, C_2, \forall R.(\exists R.A)), \exists R.B)\} \\
 & | & \\
 & |\ R & \\
 & | & \\
x_1 & \bullet & \{B, \exists R.A, ((\forall R.\neg A) \sqcap (\forall R.\neg B)), \forall R.\neg A, \forall R.\neg B\} \\
 & | & \\
 & |\ R & \\
 & | & \\
x_2 & \bullet & \{A\}
\end{array}
$$

Apply $\rightarrow_\forall$ to get

$$
\begin{gathered}
\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\
R(x_0, x_1), B(x_1), \exists R.A(x_1), \\
((\forall R.\neg A) \sqcap (\forall R.\neg B))(x_1), \forall R.\neg A(x_1), \forall R.\neg B(x_1), \\
R(x_1, x_2), A(x_2), \neg A(x_2)\}\}
\end{gathered}
$$

or graphically

$$
\begin{array}{lcl}
x_o & \bullet & \{C_0, C_1, C_2, \forall R.(\exists R.A)), \exists R.B)\} \\
 & | & \\
 & |\ R & \\
 & | & \\
x_1 & \bullet & \{B, \exists R.A, ((\forall R.\neg A) \sqcap (\forall R.\neg B)), \forall R.\neg A, \forall R.\neg B\} \\
 & | & \\
 & |\ R & \\
 & | & \\
x_2 & \bullet & \{A, \neg A\}
\end{array}
$$

The ABox has a clash!
There are no alternatives/other ways to apply the or rule, so C_0 is unsatisfiable.