

EECS 4401/5326 Winter 2022

Week 5 — \mathcal{ALC} Tableau Proofs Examples

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Exercise 1

Use the tableau method for \mathcal{ALC} described in Baader and Sattler's paper to check whether the following concept is satisfiable/consistent. Show the steps and rules that are used. If the concept is satisfiable give the model(s) (satisfying interpretation(s)) obtained by the method.

$$((\forall R.(\exists R.A)) \sqcap (\exists R.B)) \sqcap (\forall R.((\forall R.\neg A) \sqcup (\forall R.\neg B)))$$

Solution

Let

$$C_0 = ((\forall R.(\exists R.A)) \sqcap (\exists R.B)) \sqcap (\forall R.((\forall R.\neg A) \sqcup (\forall R.\neg B)))$$

$$C_1 = ((\forall R.(\exists R.A)) \sqcap (\exists R.B))$$

$$C_2 = (\forall R.((\forall R.\neg A) \sqcup (\forall R.\neg B)))$$

Start with

$$\mathcal{S} = \{\{C_0(x_0)\}\}$$

or graphically

$$x_o \bullet \{C_0\}$$

Apply \rightarrow_{\sqcap} to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0)\}\}$$

or graphically

$$x_o \bullet \{C_0, C_1, C_2\}$$

Apply \rightarrow_{\square} again to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0)\}\}$$

or graphically

$$x_o \bullet \{C_0, C_1, C_2, \forall R.(\exists R.A), \exists R.B\}$$

Apply \rightarrow_{\exists} to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\ R(x_0, x_1), B(x_1)\}\}$$

or graphically

$$\begin{array}{c} x_o \bullet \{C_0, C_1, C_2, \forall R.(\exists R.A), \exists R.B\} \\ | \\ R \\ | \\ x_1 \bullet \{B\} \end{array}$$

Apply \rightarrow_{\forall} to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\ R(x_0, x_1), B(x_1), \exists R.A(x_1)\}\}$$

or graphically

$$\begin{array}{c} x_o \bullet \{C_0, C_1, C_2, \forall R.(\exists R.A), \exists R.B\} \\ | \\ R \\ | \\ x_1 \bullet \{B, \exists R.A\} \end{array}$$

Apply \rightarrow_{\forall} to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), R(x_0, x_1), B(x_1), \exists R.A(x_1), ((\forall R.\neg A) \sqcup (\forall R.\neg B))(x_1)\}\}$$

or graphically

$$\begin{array}{c} x_0 \bullet \quad \{C_0, C_1, C_2, \forall R.(\exists R.A), \exists R.B\} \\ | \\ | R \\ | \\ x_1 \bullet \quad \{B, \exists R.A, ((\forall R.\neg A) \sqcup (\forall R.\neg B))\} \end{array}$$

Apply \rightarrow_{\exists} to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), R(x_0, x_1), B(x_1), \exists R.A(x_1), ((\forall R.\neg A) \sqcup (\forall R.\neg B))(x_1), R(x_1, x_2), A(x_2)\}\}$$

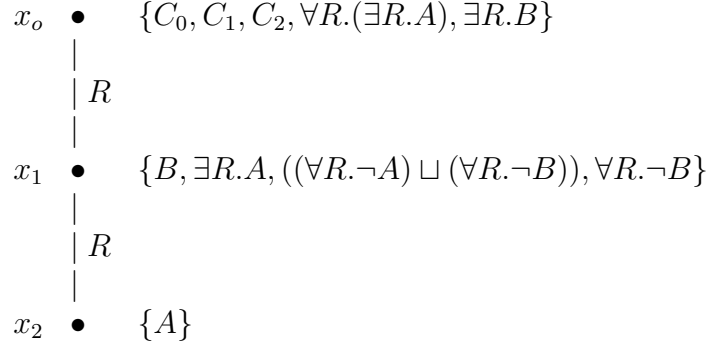
or graphically

$$\begin{array}{c} x_0 \bullet \quad \{C_0, C_1, C_2, \forall R.(\exists R.A), \exists R.B\} \\ | \\ | R \\ | \\ x_1 \bullet \quad \{B, \exists R.A, ((\forall R.\neg A) \sqcup (\forall R.\neg B))\} \\ | \\ | R \\ | \\ x_2 \bullet \quad \{A\} \end{array}$$

Apply \rightarrow_{\sqcup} to get

$$\begin{aligned} \mathcal{S} = \{ & \{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\ & R(x_0, x_1), B(x_1), \exists R.A(x_1), ((\forall R.\neg A) \sqcup (\forall R.\neg B))(x_1), \forall R.\neg B(x_1), \\ & R(x_1, x_2), A(x_2)\}, \\ & \{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\ & R(x_0, x_1), B(x_1), \exists R.A(x_1), ((\forall R.\neg A) \sqcup (\forall R.\neg B))(x_1), \forall R.\neg A(x_1), \\ & R(x_1, x_2), A(x_2)\} \} \end{aligned}$$

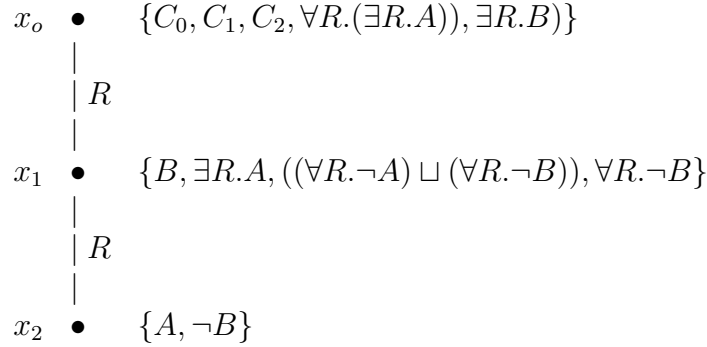
or graphically for the first ABox



Apply \rightarrow_{\forall} to get

$$\begin{aligned}
 \mathcal{S} = \{ & \{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\
 & R(x_0, x_1), B(x_1), \exists R.A(x_1), ((\forall R.\neg A) \sqcup (\forall R.\neg B))(x_1), \forall R.\neg B(x_1), \\
 & R(x_1, x_2), A(x_2), \neg B(x_2)\}, \\
 & \{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\
 & R(x_0, x_1), B(x_1), \exists R.A(x_1), ((\forall R.\neg A) \sqcup (\forall R.\neg B))(x_1), \forall R.\neg A(x_1), \\
 & R(x_1, x_2), A(x_2)\} \}
 \end{aligned}$$

or graphically for the first ABox



No clash and no further rules apply to the first ABox so C_0 is satisfiable.

A model I is $\Delta^I = \{x_0, x_1, x_2\}$, $A^I = \{x_2\}$, $B^I = \{x_1\}$, and $R^I = \{\langle x_0, x_1 \rangle, \langle x_1, x_2 \rangle\}$.

We can continue and apply \rightarrow_{\forall} to the second ABox to get

$$\mathcal{S} = \{ \{ C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\ R(x_0, x_1), B(x_1), \exists R.A(x_1), ((\forall R.\neg A) \sqcup (\forall R.\neg B))(x_1), \forall R.\neg B(x_1), \\ R(x_1, x_2), A(x_2), \neg B(x_2) \}, \\ \{ C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\ R(x_0, x_1), B(x_1), \exists R.A(x_1), ((\forall R.\neg A) \sqcup (\forall R.\neg B))(x_1), \forall R.\neg A(x_1), \\ R(x_1, x_2), A(x_2), \neg A(x_2) \} \}$$

or graphically for the second ABox

$$\begin{array}{l} x_0 \bullet \quad \{ C_0, C_1, C_2, \forall R.(\exists R.A), \exists R.B \} \\ | \\ | R \\ | \\ x_1 \bullet \quad \{ B, \exists R.A, ((\forall R.\neg A) \sqcup (\forall R.\neg B)), \forall R.\neg A \} \\ | \\ | R \\ | \\ x_2 \bullet \quad \{ A, \neg A \} \end{array}$$

There is a clash so this ABox is unsatisfiable. No further rules apply so there are no further models.

Exercise 2

Use the tableau method for \mathcal{ALC} described in Baader and Sattler's paper to check whether the following concept is satisfiable/consistent. Show the steps and rules that are used. If the concept is satisfiable give the model(s) (satisfying interpretation(s)) obtained by the method.

$$((\forall R.(\exists R.A)) \sqcap (\exists R.B)) \sqcap (\forall R.((\forall R.\neg A) \sqcap (\forall R.\neg B)))$$

Solution

Let

$$\begin{aligned} C_0 &= ((\forall R.(\exists R.A)) \sqcap (\exists R.B)) \sqcap (\forall R.((\forall R.\neg A) \sqcap (\forall R.\neg B))) \\ C_1 &= ((\forall R.(\exists R.A)) \sqcap (\exists R.B)) \\ C_2 &= (\forall R.((\forall R.\neg A) \sqcap (\forall R.\neg B))) \end{aligned}$$

Start with

$$\mathcal{S} = \{\{C_0(x_0)\}\}$$

or graphically

$$x_o \bullet \{C_0\}$$

Apply \rightarrow_{\sqcap} to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0)\}\}$$

or graphically

$$x_o \bullet \{C_0, C_1, C_2\}$$

Apply \rightarrow_{\sqcap} again to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0)\}\}$$

or graphically

$$x_o \bullet \{C_0, C_1, C_2, \forall R.(\exists R.A), \exists R.B\}$$

Apply \rightarrow_{\exists} to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), R(x_0, x_1), B(x_1)\}\}$$

or graphically

$$\begin{array}{c} x_o \bullet \{C_0, C_1, C_2, \forall R.(\exists R.A), \exists R.B\} \\ | \\ R \\ | \\ x_1 \bullet \{B\} \end{array}$$

Apply \rightarrow_{\forall} to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), R(x_0, x_1), B(x_1), \exists R.A(x_1)\}\}$$

or graphically

$$\begin{array}{c} x_o \bullet \{C_0, C_1, C_2, \forall R.(\exists R.A), \exists R.B\} \\ | \\ R \\ | \\ x_1 \bullet \{B, \exists R.A\} \end{array}$$

Apply \rightarrow_{\forall} to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), R(x_0, x_1), B(x_1), \exists R.A(x_1), ((\forall R.\neg A) \sqcap (\forall R.\neg B))(x_1)\}\}$$

or graphically

$$\begin{array}{c} x_0 \bullet \quad \{C_0, C_1, C_2, \forall R.(\exists R.A), \exists R.B\} \\ | \\ R \\ | \\ x_1 \bullet \quad \{B, \exists R.A, ((\forall R.\neg A) \sqcap (\forall R.\neg B))\} \end{array}$$

Apply \rightarrow_{\exists} to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), R(x_0, x_1), B(x_1), \exists R.A(x_1), ((\forall R.\neg A) \sqcap (\forall R.\neg B))(x_1), R(x_1, x_2), A(x_2)\}\}$$

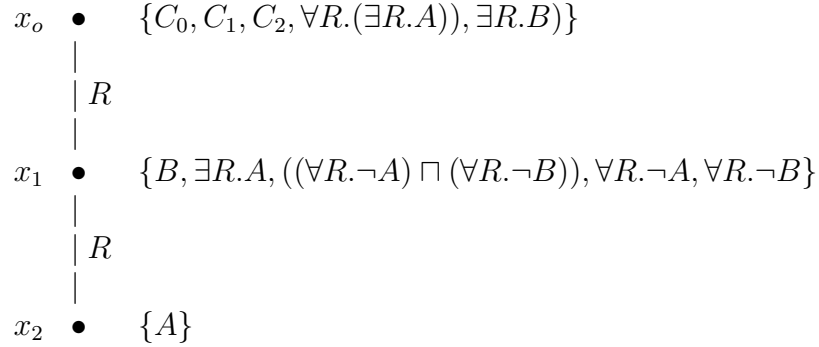
or graphically

$$\begin{array}{c} x_0 \bullet \quad \{C_0, C_1, C_2, \forall R.(\exists R.A), \exists R.B\} \\ | \\ R \\ | \\ x_1 \bullet \quad \{B, \exists R.A, ((\forall R.\neg A) \sqcap (\forall R.\neg B))\} \\ | \\ R \\ | \\ x_2 \bullet \quad \{A\} \end{array}$$

Apply \rightarrow_{\sqcap} again to get

$$\mathcal{S} = \{\{C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), R(x_0, x_1), B(x_1), \exists R.A(x_1), ((\forall R.\neg A) \sqcap (\forall R.\neg B))(x_1), \forall R.\neg A(x_1), \forall R.\neg B(x_1), R(x_1, x_2), A(x_2)\}\}$$

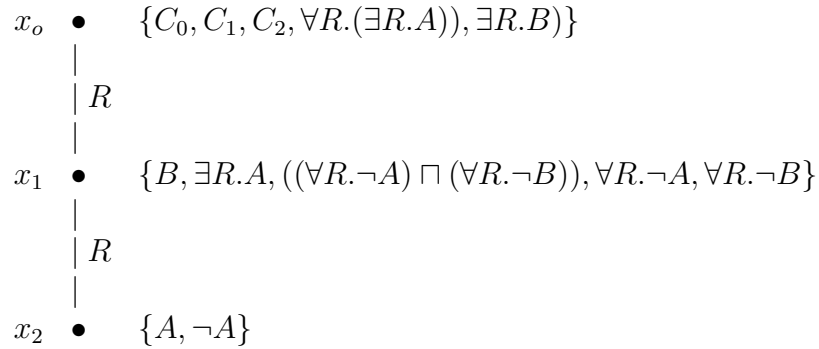
or graphically



Apply \rightarrow_{\forall} to get

$$\begin{aligned}
 \mathcal{S} = \{ \{ & C_0(x_0), C_1(x_0), C_2(x_0), \forall R.(\exists R.A)(x_0), \exists R.B(x_0), \\
 & R(x_0, x_1), B(x_1), \exists R.A(x_1), \\
 & ((\forall R.\neg A) \sqcap (\forall R.\neg B))(x_1), \forall R.\neg A(x_1), \forall R.\neg B(x_1), \\
 & R(x_1, x_2), A(x_2), \neg A(x_2) \} \}
 \end{aligned}$$

or graphically



The ABox has a clash!

There are no alternatives/other ways to apply the or rule, so C_0 is unsatisfiable.