EECS 4422/5323 Computer Vision

Unit 2: Image Representation

Outline

- Introduction
- Basics
- The Fourier transform
- Local operators
- Restoration and enhancement
- The discrete case
- Local scale and orientation

Introduction: Motivation and approach

Representation is key to enabling understanding

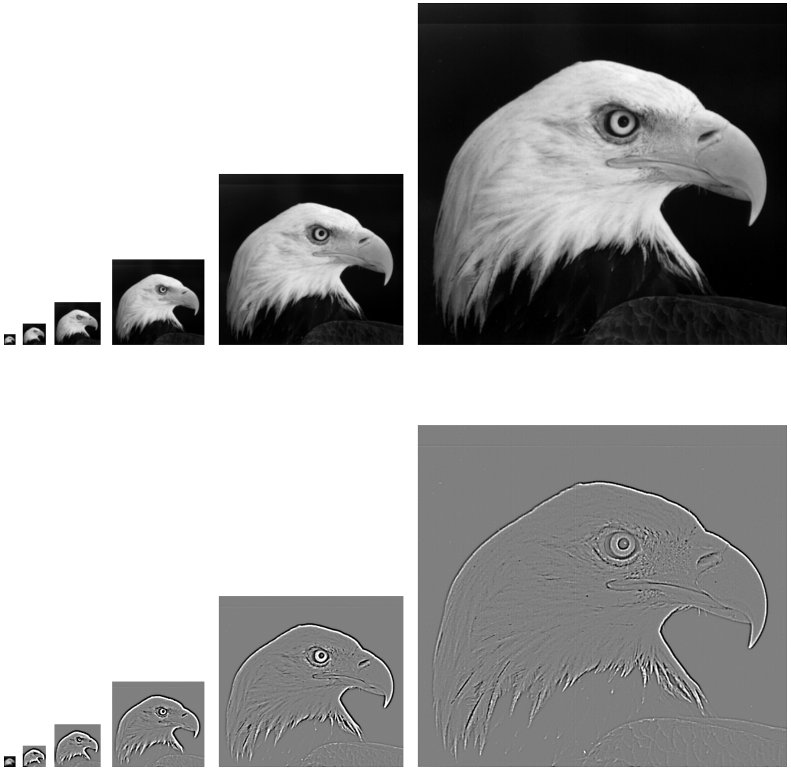
- It is often useful to transform an image in some way as a preliminary step in our analysis.
- We seek to produce a new image that is more amenable to further manipulation.

Approach

- Introduce tools with certain analytic properties to allow guidance by theory.
- Demonstrate the utility of the concept of spatial frequency.

Remarks

- Much of what we will cover in this unit would be found as part of a course on image processing.
- Moreover, many of the tools developed are straightforward extensions of those used in classical 1D signal representation/analysis.



Outline

Basics

Basics: Overview

Linear, shift invariant systems

Convolution

The point-spread function

The modulation transfer function

Basics: Overview

Linear, shift invariant systems

Convolution

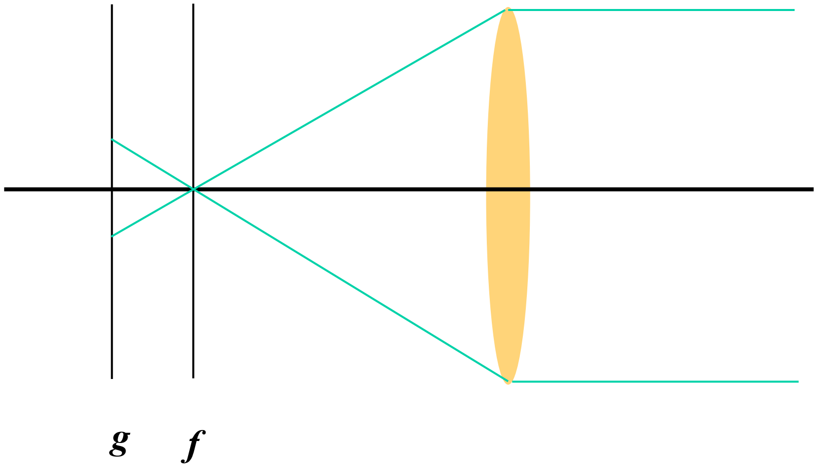
The point-spread function

The modulation transfer function

Basics: Linear, shift invariant (LSI) systems

Intuition

- Consider
  - An ideal, in focus image, f
  - Its out of focus counterpart, g
- Suppose the light is changed so that
  - the irradiance in f is doubled,
  - then so is the irradiance in g doubled.
- Suppose the imaging system is moved so that
  - the pattern in f is shifted,
  - then the pattern in g is similarly shifted.
- The transformation from the ideal to the out of focus system is said to be a linear, shift invariant operation.



Basics: Linear, shift invariant (LSI) systems

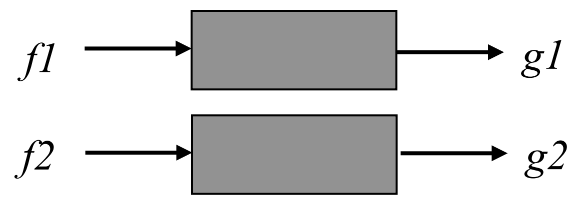
Consider a 2D system

- Let the system produce output $g1(x,y)$ when given $f1(x,y)$
- Let the system produce output $g2(x,y)$ when given $f2(x,y)$

Basics: Linear, shift invariant (LSI) systems

Consider a 2D system

- Let the system produce output $g1(x,y)$ when given $f1(x,y)$
- Let the system produce output $g2(x,y)$ when given $f2(x,y)$



A system is linear

- if the output $(a\, g1(x,y) + b\, g2(x,y))$ is produced when the input is $(a\, f1(x,y) + b\, f2(x,y))$.

Basics: Linear, shift invariant (LSI) systems

Consider a 2D system

- Let the system produce output $g1(x,y)$ when given $f1(x,y)$
- Let the system produce output $g2(x,y)$ when given $f2(x,y)$

A system is linear

- if the output $(a\, g1(x,y) + b\, g2(x,y))$ is produced when the input is $(a\, f1(x,y) + b\, f2(x,y))$.

Remarks

- Most real systems are limited in their maximum response and thus cannot be strictly linear.
- Irradiance, power/unit area, cannot be negative; so, visual input is restricted to be nonnegative.

Basics: Linear, shift invariant (LSI) systems

Consider a 2D system

- Let the system produce output $g1(x,y)$ when given $f1(x,y)$

Basics: Linear, shift invariant (LSI) systems

Consider a 2D system

- Let the system produce output $g1(x,y)$ when given $f1(x,y)$

A system is shift invariant

- If it produces the shifted output $g1(x\text{-}a,y\text{-}b)$ when given the input $f1(x\text{-}a,y\text{-}b)$

Basics: Linear, shift invariant (LSI) systems

Consider a 2D system

- Let the system produce output $g1(x,y)$ when given $f1(x,y)$

A system is shift invariant

- If it produces the shifted output $g1(x\text{-}a,y\text{-}b)$ when given the input $f1(x\text{-}a,y\text{-}b)$

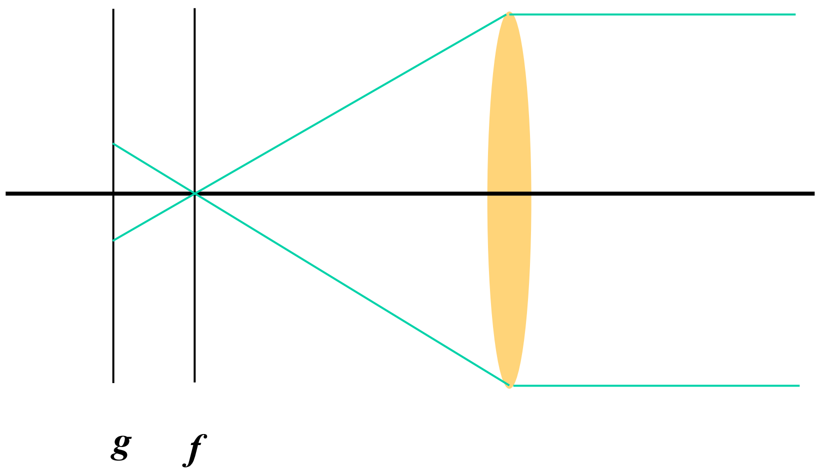
Remarks

- In practice, images are limited in area; so, shift invariance only holds for limited displacements.
- Aberrations in optical imaging vary spatially, yielding further departure from perfect shift invariance.

Basics: Linear, shift invariant (LSI) systems

Example revisited

- Consider
  - An ideal, in focus image, $f(x,y)$
  - Its out of focus counterpart, $g(x,y)$
  - The "system" is defocus.



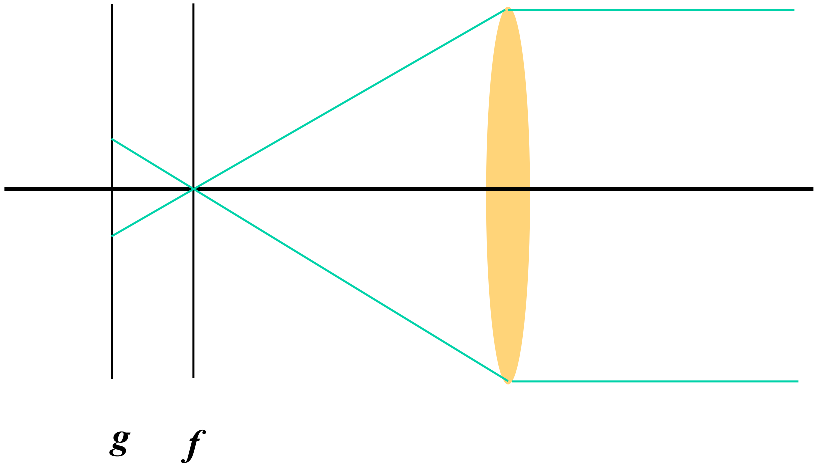
Basics: Linear, shift invariant (LSI) systems

Example revisited

- Consider
  - An ideal, in focus image, $f(x,y)$
  - Its out of focus counterpart, $g(x,y)$
  - The "system" is defocus.
- Suppose the light is changed so that
  - the irradiance in f is doubled

    → the system input is $1\ f(x,y) + 1\ f(x,y)$
  - then the irradiance in g doubled

    → the system output is $1\ g(x,y) + 1\ g(x,y)$.
  - The system is linear.



Basics: Linear, shift invariant (LSI) systems

Example revisited

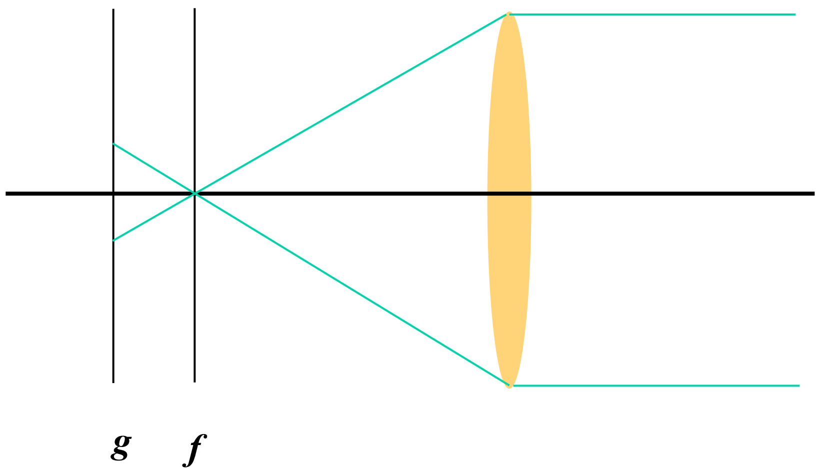
- Consider
  - An ideal, in focus image, $f(x,y)$
  - Its out of focus counterpart, $g(x,y)$
  - The "system" is defocus.
- Suppose the light is changed so that
  - the irradiance in f is doubled

    → the system input is $1\, f(x,y) + 1\, f(x,y)$
  - then the irradiance in g doubled

    → the system output is $1\, g(x,y) + 1\, g(x,y)$.
  - The system is linear.
- Suppose the imaging system is moved so that
  - the pattern in f is shifted

    → the system input is $f(x\text{-}a,\ y\text{-}b)$
  - then the pattern in g is similarly shifted

    → the system output is $g(x\text{-}a,\ y\text{-}b)$.
  - The system is shift invariant.



Basics: Overview

Linear, shift invariant systems

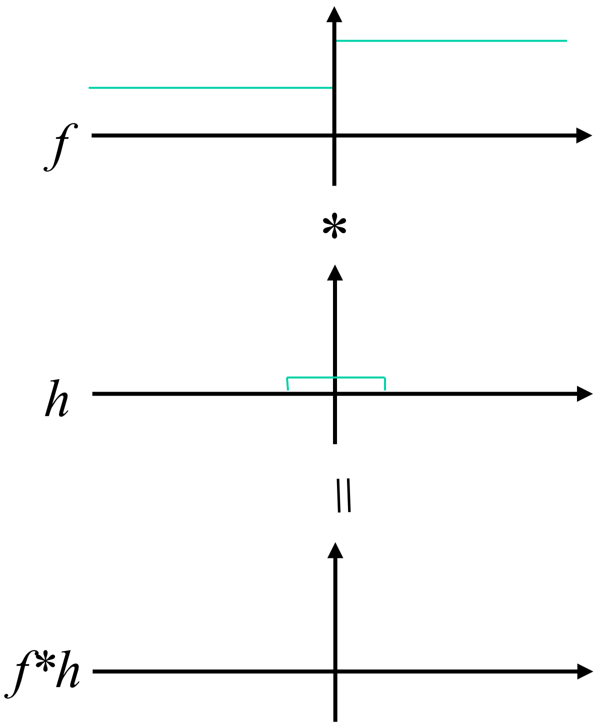
Convolution

The point-spread function

The modulation transfer function

Basics: Smoothing via local averaging

Graphic depiction



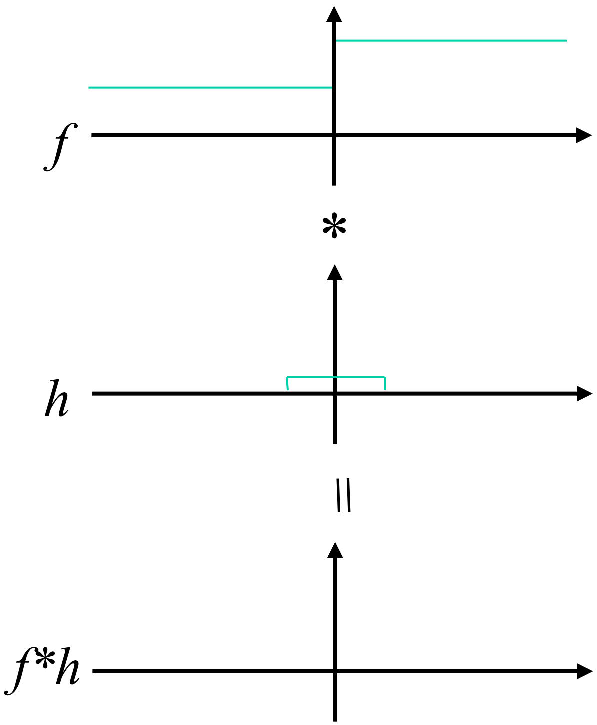
Numerical calculation

x	s	f(x-s)h(s)	+
-2	-1		
	0		
	1		
-1	-1		
	0		
	1		
0	-1		
	0		
	1		
1	-1		
	0		
	1		
2	-1		
	0		
	1		

Formalization

Basics: Smoothing via local averaging

Graphic depiction



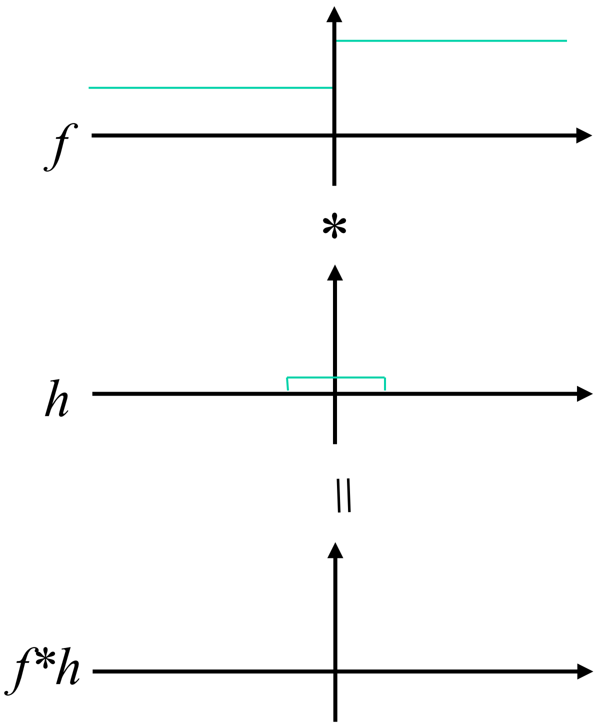
Numerical calculation

x	s	f(x-s)h(s)	+
-2	-1	(1)(1/3)	
	0		
	1		
-1	-1		
	0		
	1		
0	-1		
	0		
	1		
1	-1		
	0		
	1		
2	-1		
	0		
	1		

Formalization

Basics: Smoothing via local averaging

Graphic depiction



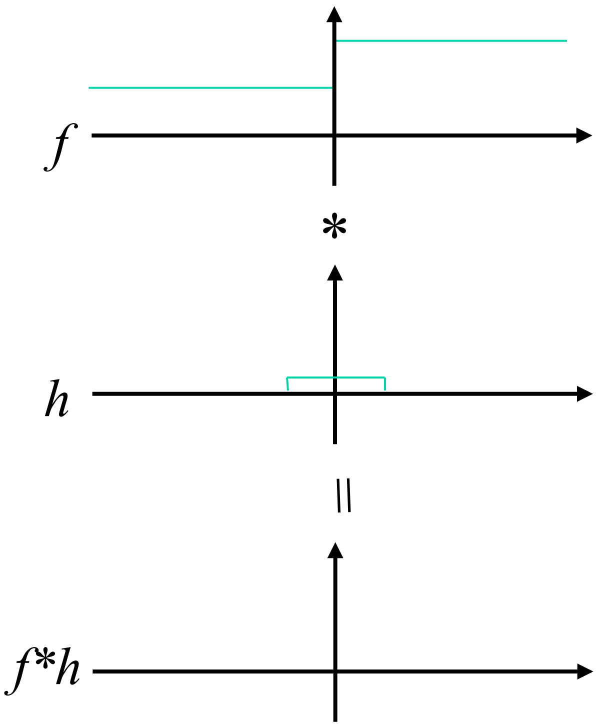
Formalization

Numerical calculation

x	s	f(x-s)h(s)	+
-2	-1	(1)(1/3)	
	0	(1)(1/3)	
	1		
-1	-1		
	0		
	1		
0	-1		
	0		
	1		
1	-1		
	0		
	1		
2	-1		
	0		
	1		

Basics: Smoothing via local averaging

Graphic depiction



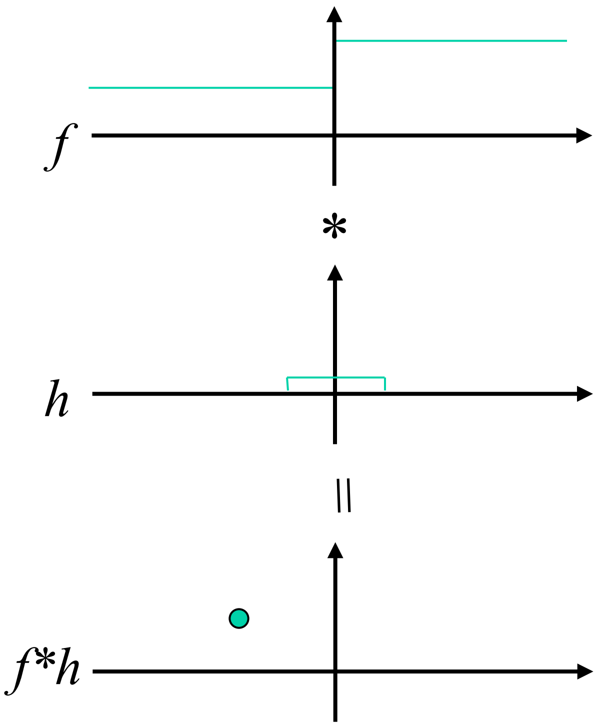
Numerical calculation

x	s	f(x-s)h(s)	+
-2	-1	(1)(1/3)	
	0	(1)(1/3)	
	1	(1)(1/3)	
-1	-1		
	0		
	1		
0	-1		
	0		
	1		
1	-1		
	0		
	1		
2	-1		
	0		
	1		

Formalization

Basics: Smoothing via local averaging

Graphic depiction



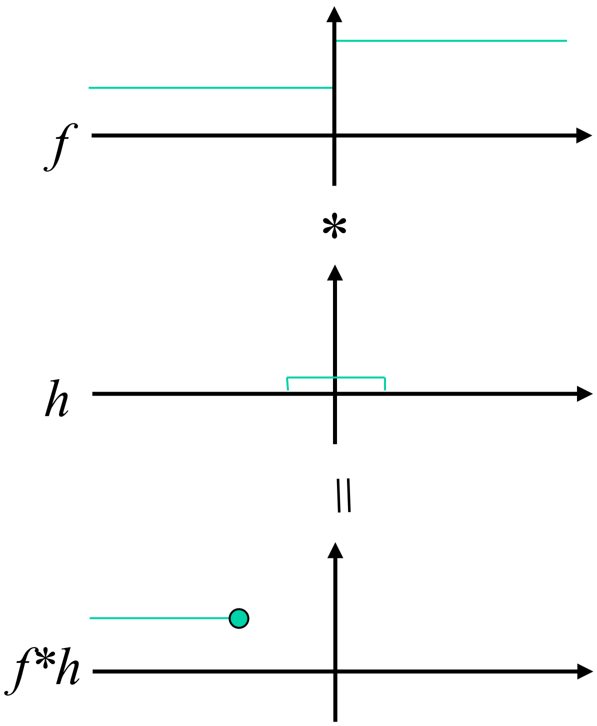
Formalization

Numerical calculation

x	s	f(x-s)h(s)	+
-2	-1	(1)(1/3)	
	0	(1)(1/3)	
	1	(1)(1/3)	1
-1	-1		
	0		
	1		
0	-1		
	0		
	1		
1	-1		
	0		
	1		
2	-1		
	0		
	1		

Basics: Smoothing via local averaging

Graphic depiction



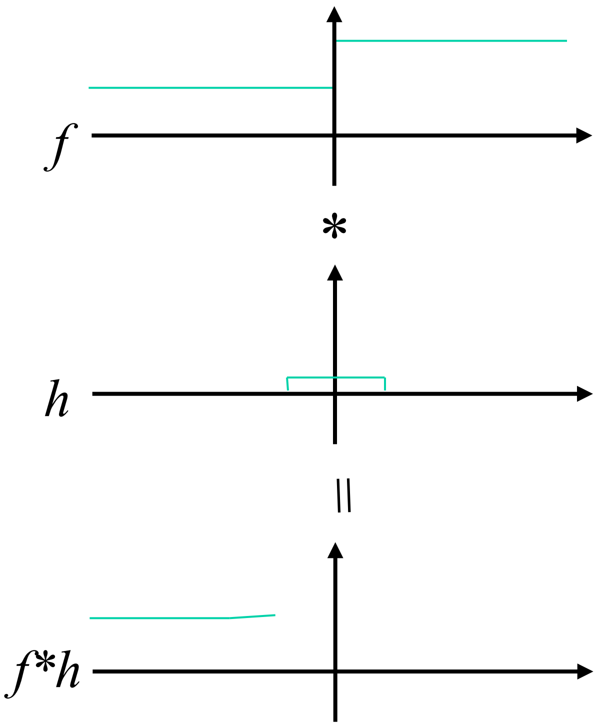
Numerical calculation

x	s	f(x-s)h(s)	+
-2	-1	(1)(1/3)	
	0	(1)(1/3)	
	1	(1)(1/3)	1
-1	-1		
	0		
	1		
0	-1		
	0		
	1		
1	-1		
	0		
	1		
2	-1		
	0		
	1		

Formalization

Basics: Smoothing via local averaging

Graphic depiction



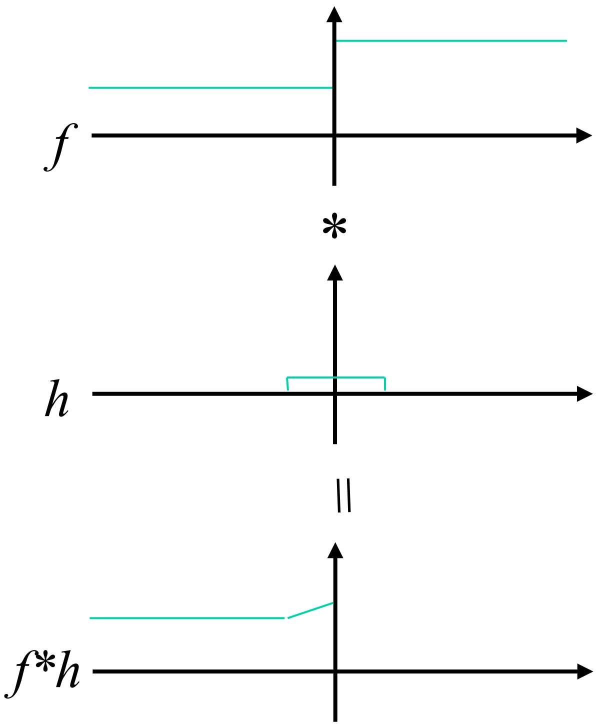
Formalization

Numerical calculation

x	s	f(x-s)h(s)	+
-2	-1	(1)(1/3)	
	0	(1)(1/3)	
	1	(1)(1/3)	1
-1	-1	(3/2)(1/3)	
	0	(1)(1/3)	
	1	(1)(1/3)	7.0/6.0
0	-1		
	0		
	1		
1	-1		
	0		
	1		
2	-1		
	0		
	1		

Basics: Smoothing via local averaging

Graphic depiction



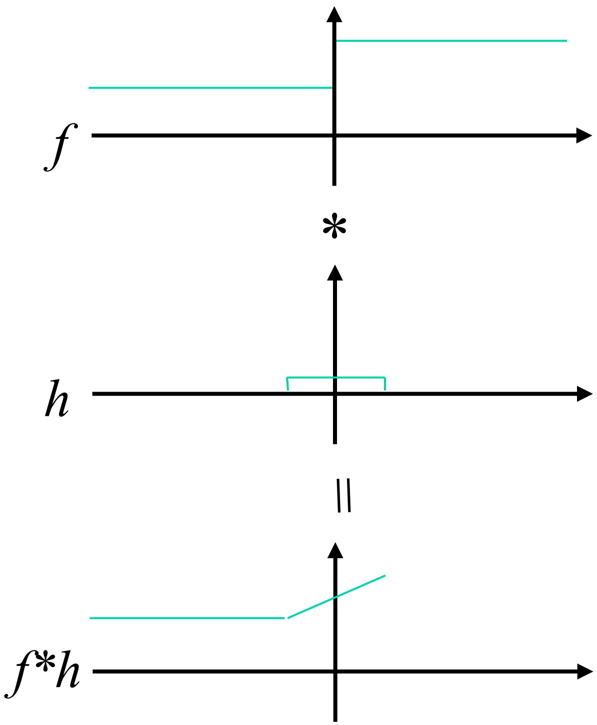
Numerical calculation

x	s	f(x-s)h(s)	+
-2	-1	(1)(1/3)	
	0	(1)(1/3)	
	1	(1)(1/3)	1
-1	-1	(3/2)(1/3)	
	0	(1)(1/3)	
	1	(1)(1/3)	7.0/6.0
0	-1	(2)(1/3)	
	0	(3/2)(1/3)	
	1	(1)(1/3)	3.0/2.0
1	-1		
	0		
	1		
2	-1		
	0		
	1		

Formalization

Basics: Smoothing via local averaging

Graphic depiction



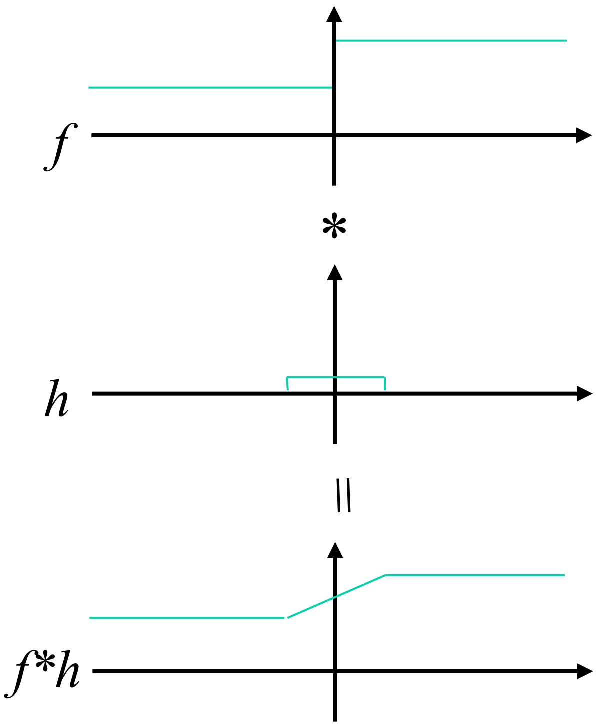
Numerical calculation

x	s	f(x-s)h(s)	+
-2	-1	(1)(1/3)	
	0	(1)(1/3)	
	1	(1)(1/3)	1
-1	-1	(3/2)(1/3)	
	0	(1)(1/3)	
	1	(1)(1/3)	7.0/6.0
0	-1	(2)(1/3)	
	0	(3/2)(1/3)	
	1	(1)(1/3)	3.0/2.0
1	-1	(2)(1/3)	
	0	(2)(1/3)	
	1	(3/2)(1/3)	11.0/6.0
2	-1		
	0		
	1		

Formalization

Basics: Smoothing via local averaging

Graphic depiction



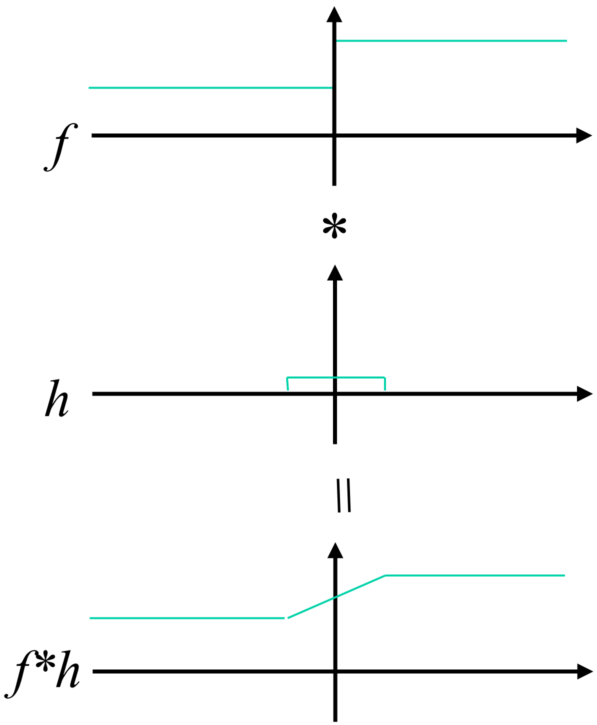
Numerical calculation

x	s	f(x-s)h(s)	+
-2	-1	(1)(1/3)	
	0	(1)(1/3)	
	1	(1)(1/3)	1
-1	-1	(3/2)(1/3)	
	0	(1)(1/3)	
	1	(1)(1/3)	7.0/6.0
0	-1	(2)(1/3)	
	0	(3/2)(1/3)	
	1	(1)(1/3)	3.0/2.0
1	-1	(2)(1/3)	
	0	(2)(1/3)	
	1	(3/2)(1/3)	11.0/6.0
2	-1	(2)(1/3)	
	0	(2)(1/3)	
	1	(2)(1/3)	2

Formalization

Basics: Smoothing via local averaging

Graphic depiction



Numerical calculation

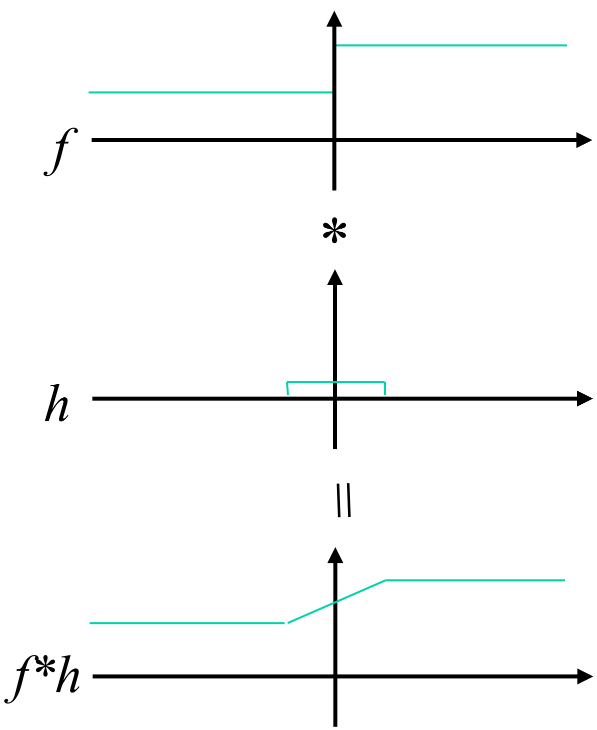
x	s	f(x-s)h(s)	+
-2	-1	(1)(1/3)	
	0	(1)(1/3)	
	1	(1)(1/3)	1
-1	-1	(3/2)(1/3)	
	0	(1)(1/3)	
	1	(1)(1/3)	7.0/6.0
0	-1	(2)(1/3)	
	0	(3/2)(1/3)	
	1	(1)(1/3)	3.0/2.0
1	-1	(2)(1/3)	
	0	(2)(1/3)	
	1	(3/2)(1/3)	11.0/6.0
2	-1	(2)(1/3)	
	0	(2)(1/3)	
	1	(2)(1/3)	2

Formalization

- We begin by considering a function: $f(x)$

Basics: Smoothing via local averaging

Graphic depiction



Numerical calculation

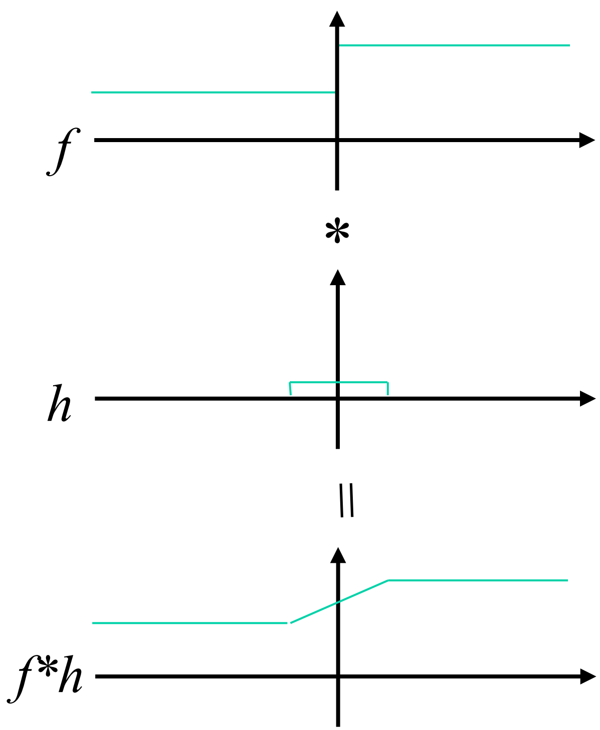
x	s	f(x-s)h(s)	+
-2	-1	(1)(1/3)	
	0	(1)(1/3)	
	1	(1)(1/3)	1
-1	-1	(3/2)(1/3)	
	0	(1)(1/3)	
	1	(1)(1/3)	7.0/6.0
0	-1	(2)(1/3)	
	0	(3/2)(1/3)	
	1	(1)(1/3)	3.0/2.0
1	-1	(2)(1/3)	
	0	(2)(1/3)	
	1	(3/2)(1/3)	11.0/6.0
2	-1	(2)(1/3)	
	0	(2)(1/3)	
	1	(2)(1/3)	2

Formalization

- And we multiply it with values of another function: $f(x)h$

Basics: Smoothing via local averaging

Graphic depiction



Numerical calculation

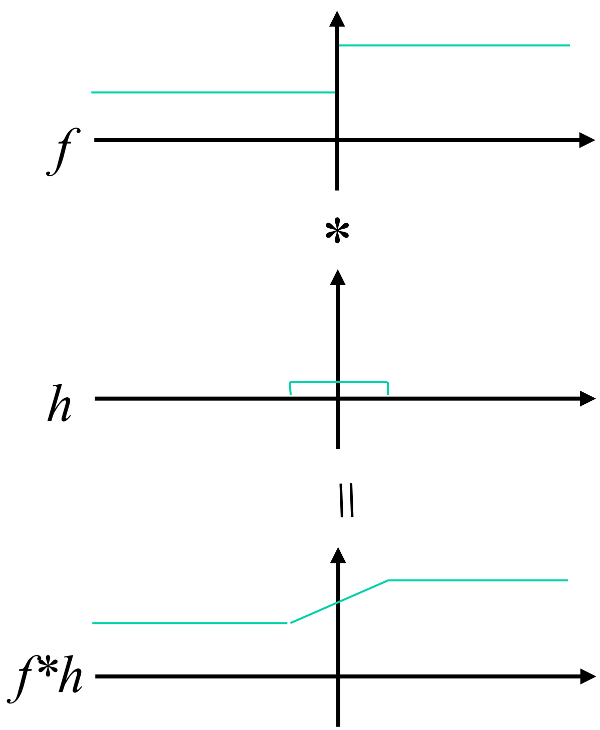
x	s	f(x-s)h(s)	+
-2	-1	(1)(1/3)	
	0	(1)(1/3)	
	1	(1)(1/3)	1
-1	-1	(3/2)(1/3)	
	0	(1)(1/3)	
	1	(1)(1/3)	7.0/6.0
0	-1	(2)(1/3)	
	0	(3/2)(1/3)	
	1	(1)(1/3)	3.0/2.0
1	-1	(2)(1/3)	
	0	(2)(1/3)	
	1	(3/2)(1/3)	11.0/6.0
2	-1	(2)(1/3)	
	0	(2)(1/3)	
	1	(2)(1/3)	2

Formalization

- But we do this at various offsets: $f(x-s)h(s)$

Basics: Smoothing via local averaging

Graphic depiction



Numerical calculation

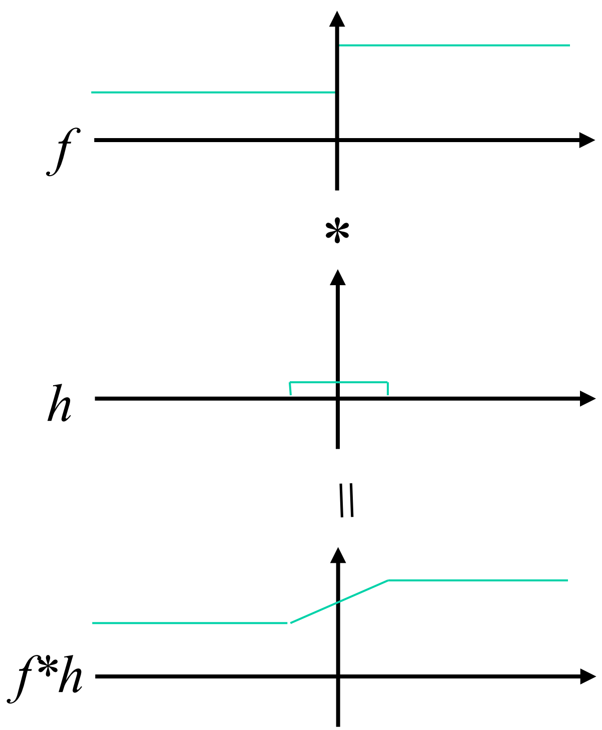
x	s	f(x-s)h(s)	+
-2	-1	(1)(1/3)	
	0	(1)(1/3)	
	1	(1)(1/3)	1
-1	-1	(3/2)(1/3)	
	0	(1)(1/3)	
	1	(1)(1/3)	7.0/6.0
0	-1	(2)(1/3)	
	0	(3/2)(1/3)	
	1	(1)(1/3)	3.0/2.0
1	-1	(2)(1/3)	
	0	(2)(1/3)	
	1	(3/2)(1/3)	11.0/6.0
2	-1	(2)(1/3)	
	0	(2)(1/3)	
	1	(2)(1/3)	2

Formalization

- and multiply by infinitesimal support elements: $f(x-s)h(s)ds$

Basics: Smoothing via local averaging

Graphic depiction



Numerical calculation

x	s	f(x-s)h(s)	+
-2	-1	(1)(1/3)	
	0	(1)(1/3)	
	1	(1)(1/3)	1
-1	-1	(3/2)(1/3)	
	0	(1)(1/3)	
	1	(1)(1/3)	7.0/6.0
0	-1	(2)(1/3)	
	0	(3/2)(1/3)	
	1	(1)(1/3)	3.0/2.0
1	-1	(2)(1/3)	
	0	(2)(1/3)	
	1	(3/2)(1/3)	11.0/6.0
2	-1	(2)(1/3)	
	0	(2)(1/3)	
	1	(2)(1/3)	2

Formalization

- Finally, we sum up (integrate): $\int f(x-s)h(s)ds$

Basics: Convolution

Definition

- Consider a system that, given $f(x,y)$ as input, produces output

$$g(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)h(\xi,\eta)d\xi d\eta$$

- We say that g is the convolution of f and h, written as $g=f*h$.

Basics: Convolution

Definition

- Consider a system that, given $f(x,y)$ as input, produces output

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)h(\xi,\eta)d\xi d\eta$$

- We say that g is the convolution of f and h, written as $g=f*h$.

Convolution is linear

- Applying the system to $(a\,f1(x,y) + b\,f2(x,y))$ yields $(a\,g1(x,y) + b\,g2(x,y))$.
- Follows from rule for integrating the product of a constant and a function
- and the rule for integrating the sum of two functions.

$$\int\left[a\alpha(\xi) + b\beta(\xi)\right]d\xi = a\int\alpha(\xi)d\xi + b\int\beta(\xi)d\xi$$

Basics: Convolution

Definition

- Consider a system that, given $f(x,y)$ as input, produces output

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)h(\xi,\eta)d\xi d\eta$$

- We say that g is the convolution of f and h, written as $g=f*h$.

Convolution is linear

- Applying the system to $(a\, f1(x,y) + b\, f2(x,y))$ yields $(a\, g1(x,y) + b\, g2(x,y))$.
- Follows from rule for integrating the product of a constant and a function
- and the rule for integrating the sum of two functions.

Convolution is shift invariant

- Applying the system to $f(x\text{-}a,y\text{-}b)$ yields $g(x\text{-}a,y\text{-}b)$.
- Follows from the convolution integral being independent of (x,y)
- So a change of variables $(x,y) \rightarrow (x\text{-}a,y\text{-}b)=(x',y')$ just shifts the result.

Basics: Convolution

Definition

- Consider a system that, given $f(x,y)$ as input, produces output

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)h(\xi,\eta)d\xi d\eta$$

- We say that g is the convolution of f and $h,$ written as $g=f*h$.

Convolution is linear

- Applying the system to $(a\, f1(x,y) + b\, f2(x,y))$ yields $(a\, g1(x,y) + b\, g2(x,y))$.
- Follows from rule for integrating the product of a constant and a function
- and the rule for integrating the sum of two functions.

Convolution is shift invariant

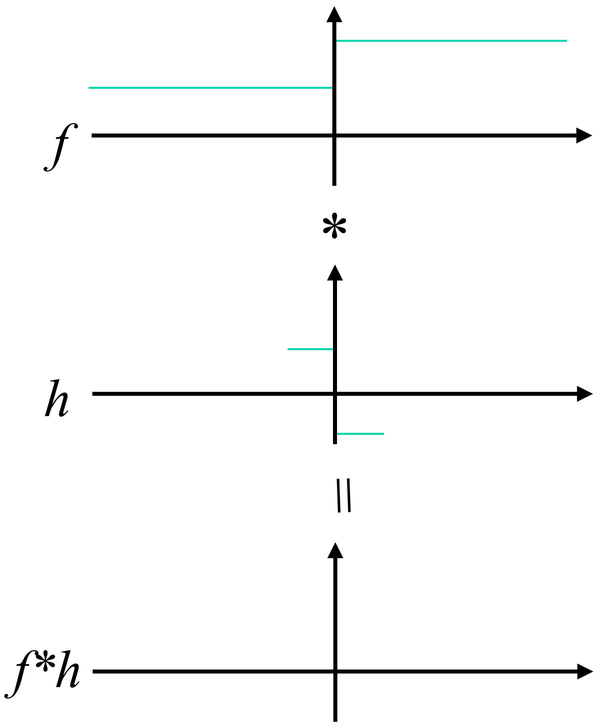
- Applying the system to $f(x\text{-}a,y\text{-}b)$ yields $g(x\text{-}a,y\text{-}b)$.
- Follows from the convolution integral being independent of (x,y)
- So a change of variables $(x,y) \rightarrow (x\text{-}a,y\text{-}b)=(x',y')$ just shifts the result.

The converse also is true

- Any linear shift invariant system performs a convolution.

Basics: Another convolution example

Graphic depiction



Numerical calculation

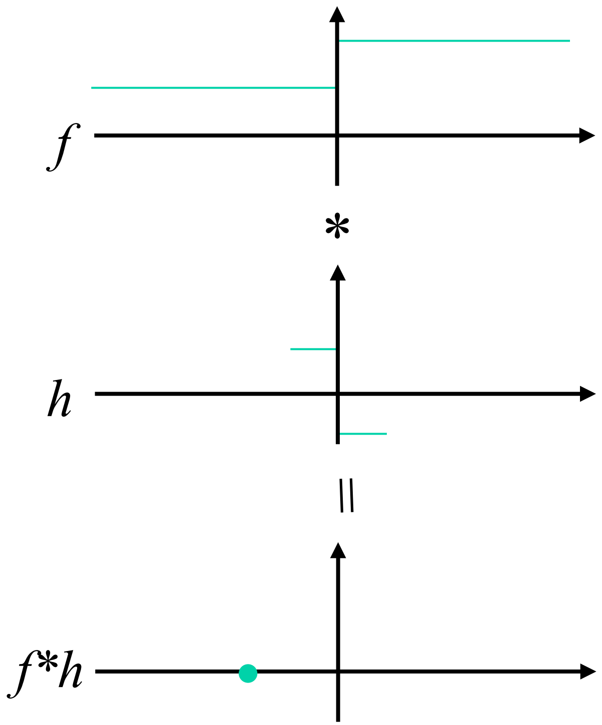
x	s	f(x-s)h(s)	+
-2	-1		
	0		
	1		
-1	-1		
	0		
	1		
0	-1		
	0		
	1		
1	-1		
	0		
	1		
2	-1		
	0		
	1		

Formalization

$$\int_{-\infty}^{\infty} f(x-s)h(s)ds$$

Basics: Another convolution example

Graphic depiction



Numerical calculation

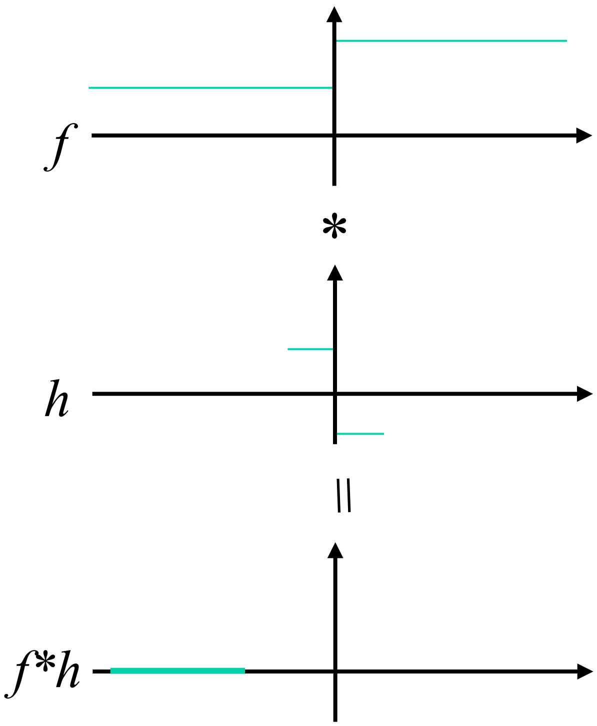
x	s	f(x-s)h(s)	+
-2	-1	(1)(1)	
	0	(1)(0)	
	1	(1)(-1)	0
-1	-1		
	0		
	1		
0	-1		
	0		
	1		
1	-1		
	0		
	1		
2	-1		
	0		
	1		

Formalization

$$\int_{-\infty}^{\infty} f(x-s)h(s)ds$$

Basics: Another convolution example

Graphic depiction



Numerical calculation

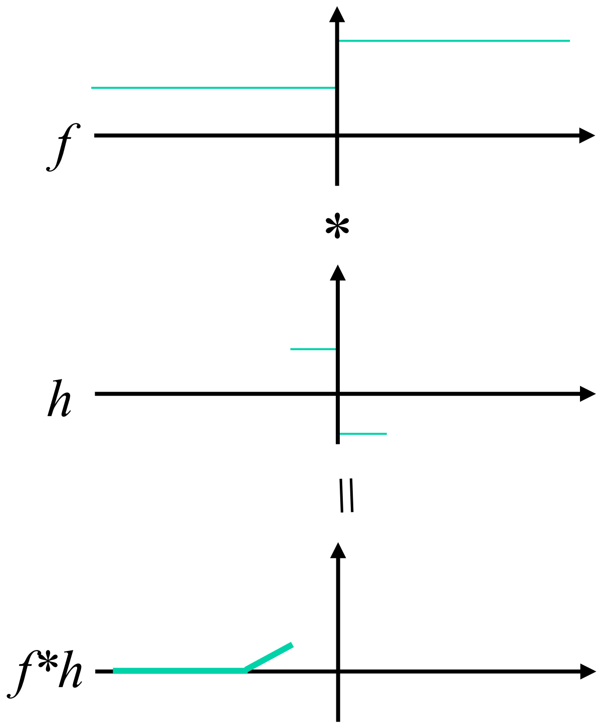
x	s	f(x-s)h(s)	+
-2	-1	(1)(1)	
	0	(1)(0)	
	1	(1)(-1)	0
-1	-1		
	0		
	1		
0	-1		
	0		
	1		
1	-1		
	0		
	1		
2	-1		
	0		
	1		

Formalization

$$\int_{-\infty}^{\infty} f(x-s)h(s)ds$$

Basics: Another convolution example

Graphic depiction



Numerical calculation

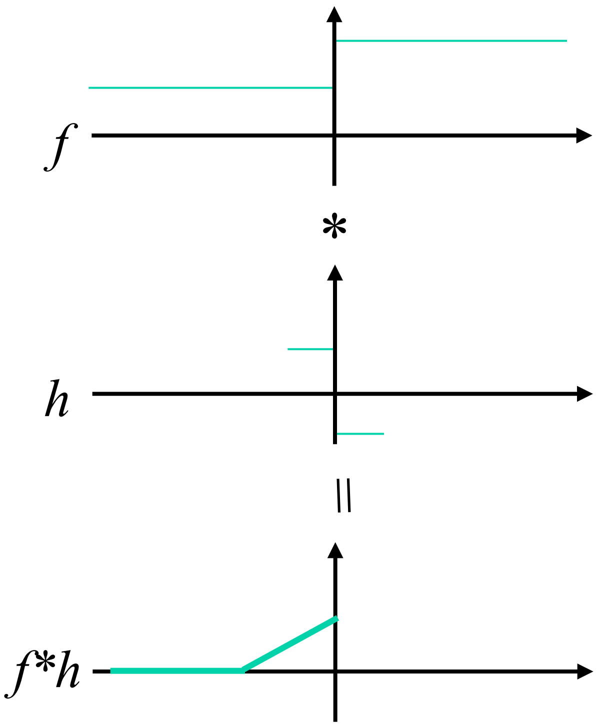
x	s	f(x-s)h(s)	+
-2	-1	(1)(1)	
	0	(1)(0)	
	1	(1)(-1)	0
-1	-1	(3/2)(1)	
	0	(1)(0)	
	1	(1)(-1)	1/2
0	-1		
	0		
	1		
1	-1		
	0		
	1		
2	-1		
	0		
	1		

Formalization

$$\int_{-\infty}^{\infty} f(x-s)h(s)ds$$

Basics: Another convolution example

Graphic depiction



Numerical calculation

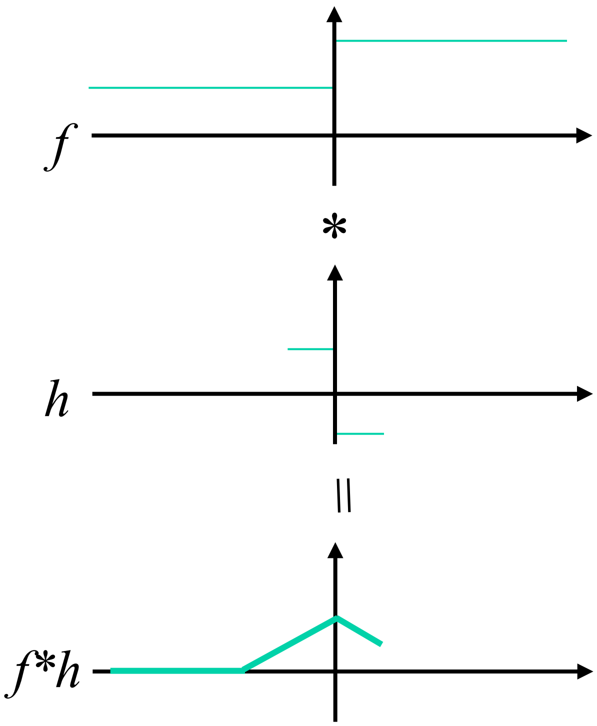
x	s	f(x-s)h(s)	+
-2	-1	(1)(1)	
	0	(1)(0)	
	1	(1)(-1)	0
-1	-1	(3/2)(1)	
	0	(1)(0)	
	1	(1)(-1)	1/2
0	-1	(2)(1)	
	0	(3/2)(0)	
	1	(1)(-1)	1
1	-1		
	0		
	1		
2	-1		
	0		
	1		

Formalization

$$\int_{-\infty}^{\infty} f(x-s)h(s)ds$$

Basics: Another convolution example

Graphic depiction



Numerical calculation

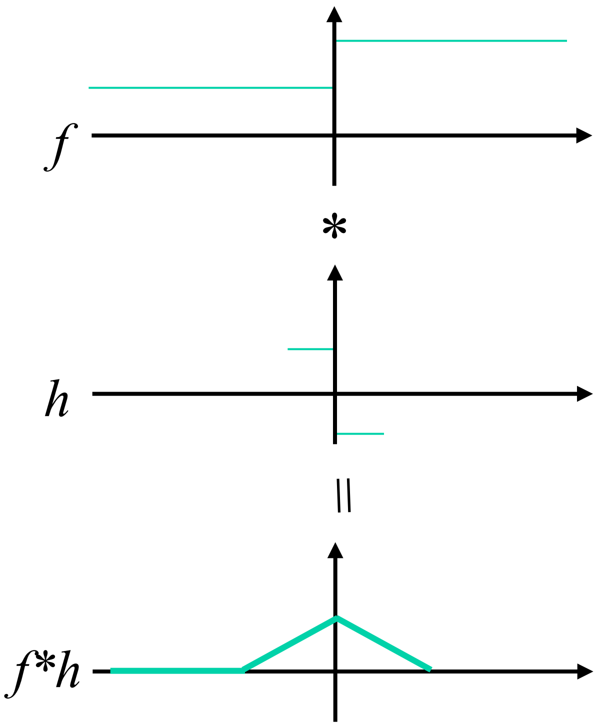
x	s	f(x-s)h(s)	+
-2	-1	(1)(1)	
	0	(1)(0)	
	1	(1)(-1)	0
-1	-1	(3/2)(1)	
	0	(1)(0)	
	1	(1)(-1)	1/2
0	-1	(2)(1)	
	0	(3/2)(0)	
	1	(1)(-1)	1
1	-1	(2)(1)	
	0	(2)(0)	
	1	(3/2)(-1)	1/2
2	-1		
	0		
	1		

Formalization

$$\int_{-\infty}^{\infty} f(x-s)h(s)ds$$

Basics: Another convolution example

Graphic depiction



Numerical calculation

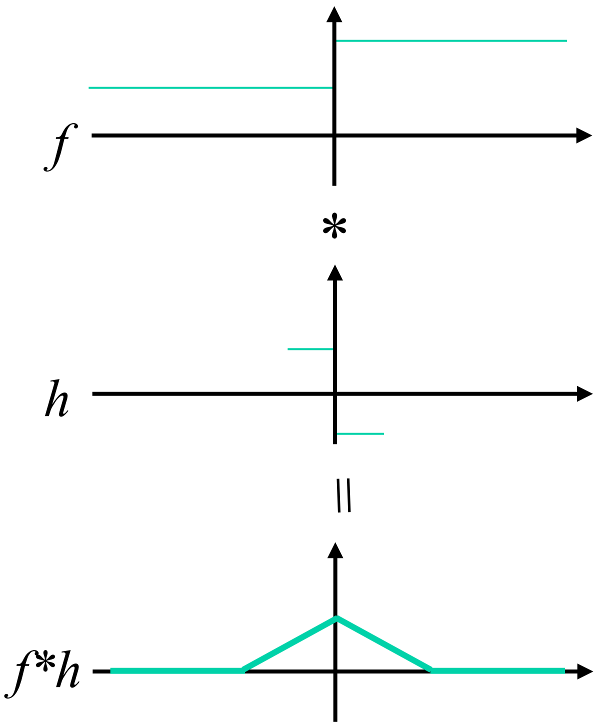
x	s	f(x-s)h(s)	+
-2	-1	(1)(1)	
	0	(1)(0)	
	1	(1)(-1)	0
-1	-1	(3/2)(1)	
	0	(1)(0)	
	1	(1)(-1)	1/2
0	-1	(2)(1)	
	0	(3/2)(0)	
	1	(1)(-1)	1
1	-1	(2)(1)	
	0	(2)(0)	
	1	(3/2)(-1)	1/2
2	-1	(2)(1)	
	0	(2)(0)	
	1	(2)(-1)	0

Formalization

$$\int_{-\infty}^{\infty} f(x-s)h(s)ds$$

Basics: Another convolution example

Graphic depiction



Formalization

$$\int_{-\infty}^{\infty} f(x-s)h(s)ds$$

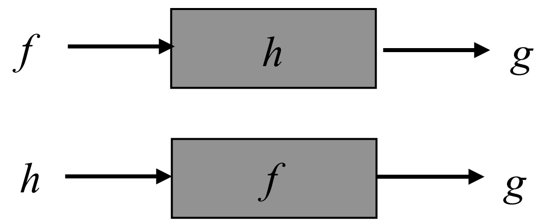
Numerical calculation

x	s	f(x-s)h(s)	+
-2	-1	(1)(1)	
	0	(1)(0)	
	1	(1)(-1)	0
-1	-1	(3/2)(1)	
	0	(1)(0)	
	1	(1)(-1)	1/2
0	-1	(2)(1)	
	0	(3/2)(0)	
	1	(1)(-1)	1
1	-1	(2)(1)	
	0	(2)(0)	
	1	(3/2)(-1)	1/2
2	-1	(2)(1)	
	0	(2)(0)	
	1	(2)(-1)	0

Basics: More fun (?) facts about convolution

Convolution is commutative

- That is $a*b = b*a$



Basics: More fun (?) facts about convolution

Convolution is commutative

- By definition

$$f(x,y)*h(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x-\xi,y-\eta)h(\xi,\eta)d\xi d\eta$$

Basics: More fun (?) facts about convolution

Convolution is commutative

- By definition

$$f(x,y)*h(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x-\xi,y-\eta)h(\xi,\eta)d\xi d\eta$$

- Consider the change of variables (so that h "appears shifted" rather than f)

$u = x-\xi \quad \xi = x-u \quad d\xi = -du$, since x is constant here

Basics: More fun (?) facts about convolution

Convolution is commutative

- By definition

$$f(x,y)*h(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi,y-\eta)h(\xi,\eta)\,d\xi d\eta$$

- Consider the change of variables (so that h "appears shifted" rather than f)

$u = x-\xi \quad \xi = x-u \quad d\xi = -du$, since x is constant here

$v = y-\eta \quad \eta = y-v \quad d\eta = -dv$, since y is constant here

Basics: More fun (?) facts about convolution

Convolution is commutative

- By definition

$$f(x,y)*h(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)h(\xi,\eta)d\xi d\eta$$

- Consider the change of variables (so that h "appears shifted" rather than f)

$u = x - \xi \quad \xi = x - u \quad d\xi = -du$, since x is constant here

$v = y - \eta \quad \eta = y - v \quad d\eta = -dv$, since y is constant here

as $\xi \to \infty, u \to -\infty \quad \xi \to -\infty, u \to \infty$

as $\eta \to \infty, v \to -\infty \quad \eta \to -\infty, v \to \infty$

Basics: More fun (?) facts about convolution

Convolution is commutative

- By definition

$$f(x,y)*h(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi,y-\eta)h(\xi,\eta)d\xi d\eta$$

- Consider the change of variables (so that h "appears shifted" rather than f)

$$u=x-\xi \quad \xi=x-u \quad d\xi=-du, \text{ since } x \text{ is constant here}$$

$$v=y-\eta \quad \eta=y-v \quad d\eta=-dv, \text{ since y is constant here}$$

$$\text{as } \xi\to\infty, u\to-\infty \quad \xi\to-\infty, u\to\infty$$

$$\text{as } \eta\to\infty, v\to-\infty \quad \eta\to-\infty, v\to\infty$$

which yields

$$f(x,y)*h(x,y)=\int_{\infty}^{-\infty}\int_{\infty}^{-\infty} f(u,v)h(x-u,y-v)(-du)(-dv)$$

Basics: More fun (?) facts about convolution

Convolution is commutative

- By definition

$$f(x,y) * h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta) h(\xi,\eta) d\xi d\eta$$

- Consider the change of variables (so that h "appears shifted" rather than f)

$$u = x - \xi \quad \xi = x - u \quad d\xi = -du, \text{ since } x \text{ is constant here}$$

$$v = y - \eta \quad \eta = y - v \quad d\eta = -dv, \text{ since y is constant here}$$

$$\text{as } \xi \to \infty, u \to -\infty \quad \xi \to -\infty, u \to \infty$$

$$\text{as } \eta \to \infty, v \to -\infty \quad \eta \to -\infty, v \to \infty$$

which yields

$$f(x,y) * h(x,y) = \int_{\infty}^{-\infty}\int_{\infty}^{-\infty} f(u,v) h(x-u, y-v)(\cancel{-}du)(\cancel{-}dv)$$

Basics: More fun (?) facts about convolution

Convolution is commutative

- We have

$$f(x,y) * h(x,y) = \int_{\infty}^{-\infty}\int_{\infty}^{-\infty} f(u,v)h(x-u,y-v)\,du\,dv$$

Basics: More fun (?) facts about convolution

Convolution is commutative

- We have

$$f(x,y) * h(x,y) = \int_{\infty}^{-\infty} \int_{\infty}^{-\infty} f(u,v)h(x-u, y-v)\,du\,dv$$

- Recall that

$$\int_{\infty}^{-\infty} [\quad] = -\int_{-\infty}^{\infty} [\quad]$$

Basics: More fun (?) facts about convolution

Convolution is commutative

- We have

$$f(x,y) * h(x,y) = \int_{\infty}^{-\infty} \int_{\infty}^{-\infty} f(u,v)h(x-u, y-v)dudv$$

- Recall that

$$\int_{\infty}^{-\infty} [\ \ ] = -\int_{-\infty}^{\infty} [\ \ ]$$

- So we can write

$$f(x,y) * h(x,y) = -\int_{-\infty}^{\infty} \left[-\int_{-\infty}^{\infty} f(u,v)h(x-u, y-v)dudv \right]$$

Basics: More fun (?) facts about convolution

Convolution is commutative

- We have

$$f(x,y)*h(x,y)=\int_{\infty}^{-\infty}\int_{\infty}^{-\infty} f(u,v)h(x-u,y-v)dudv$$

- Recall that

$$\int_{\infty}^{-\infty}[\ ]=-\int_{-\infty}^{\infty}[\ ]$$

- So we can write

$$f(x,y)*h(x,y)=\cancel{-}\int_{-\infty}^{\infty}\left[\cancel{-}\int_{-\infty}^{\infty} f(u,v)h(x-u,y-v)dudv\right]$$

Basics: More fun (?) facts about convolution

Convolution is commutative

- We have

$$f(x,y) * h(x,y) = \int_{\infty}^{-\infty} \int_{\infty}^{-\infty} f(u,v) h(x-u, y-v) du dv$$

- Recall that

$$\int_{\infty}^{-\infty} [\ ] = -\int_{-\infty}^{\infty} [\ ]$$

- So we can write

$$f(x,y) * h(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u,v) h(x-u, y-v) du dv$$

Basics: More fun (?) facts about convolution

Convolution is commutative

- We have

$$f(x,y) * h(x,y) = \int_{\infty}^{-\infty} \int_{\infty}^{-\infty} f(u,v)h(x-u, y-v)dudv$$

- Recall that

$$\int_{\infty}^{-\infty} [\ \ ] = -\int_{-\infty}^{\infty} [\ \ ]$$

- So we can write

$$f(x,y) * h(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(u,v)h(x-u, y-v)dudv$$

- Since the order of multiplication inside the integral does not matter we rewrite as

$$f(x,y) * h(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x-u, y-v) f(u,v)dudv$$

Basics: More fun (?) facts about convolution

Convolution is commutative

- We have

$$f(x,y)*h(x,y) = \int_{\infty}^{-\infty}\int_{\infty}^{-\infty} f(u,v)h(x-u,y-v)dudv$$

- Recall that

$$\int_{\infty}^{-\infty}[\ ] = -\int_{-\infty}^{\infty}[\ ]$$

- So we can write

$$f(x,y)*h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(u,v)h(x-u,y-v)dudv$$

- Since the order of multiplication inside the integral does not matter we rewrite as

$$f(x,y)*h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x-u,y-v)f(u,v)dudv$$

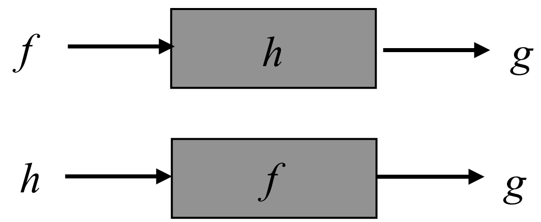
- By the definition of convolution, we conclude that

$$f(x,y)*h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x-u,y-v)f(u,v)dudv = h(x,y)*f(x,y)$$

Basics: More fun (?) facts about convolution

Convolution is commutative

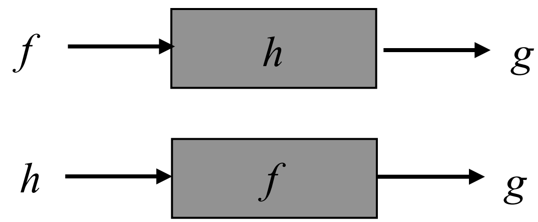
- That is $a*b = b*a$



Basics: More fun (?) facts about convolution

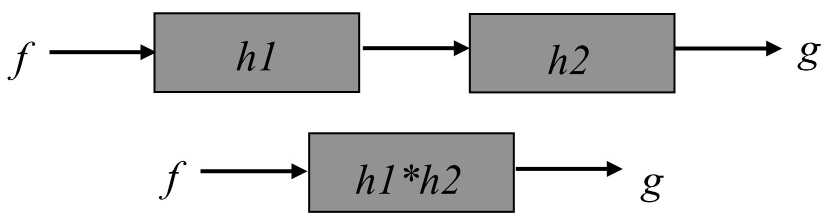
Convolution is commutative

- That is $a*b = b*a$



Convolution is associative

- That is $(a*b)*c = a*(b*c)$



Basics: Overview

Linear, shift invariant systems

Convolution

The point-spread function

The modulation transfer function

Basics: The point spread function

Relate h to an observable

- Given arbitrary $h(x,y)$, can we always find an $f(x,y)$ that produces $h(x,y)$ as output?

$$h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)h(\xi,\eta)d\xi d\eta$$

Basics: The point spread function

Relate *h* to an observable

- Given arbitrary $h(x,y)$, can we always find an $f(x,y)$ that produces $h(x,y)$ as output?

$$h(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi,y-\eta)h(\xi,\eta)d\xi d\eta$$

- Apparently, the desired f must be zero at all points, except the origin.

Basics: The point spread function

Relate h to an observable

- Given arbitrary $h(x,y)$, can we always find an $f(x,y)$ that produces $h(x,y)$ as output?

$$h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)h(\xi,\eta)d\xi d\eta$$

- Apparently, the desired f must be zero at all points, except the origin.
- Further, we will let the integral of this function over any region including the origin be 1.

Basics: The point spread function

Relate h to an observable

- Given arbitrary $h(x,y)$, can we always find an $f(x,y)$ that produces $h(x,y)$ as output?

$$h(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)h(\xi,\eta)d\xi d\eta$$

- Apparently, the desired f must be zero at all points, except the origin.
- Further, we will let the integral of this function over any region including the origin be 1.
- We call the desired function the unit impulse and denote it as $\delta(x,y)$

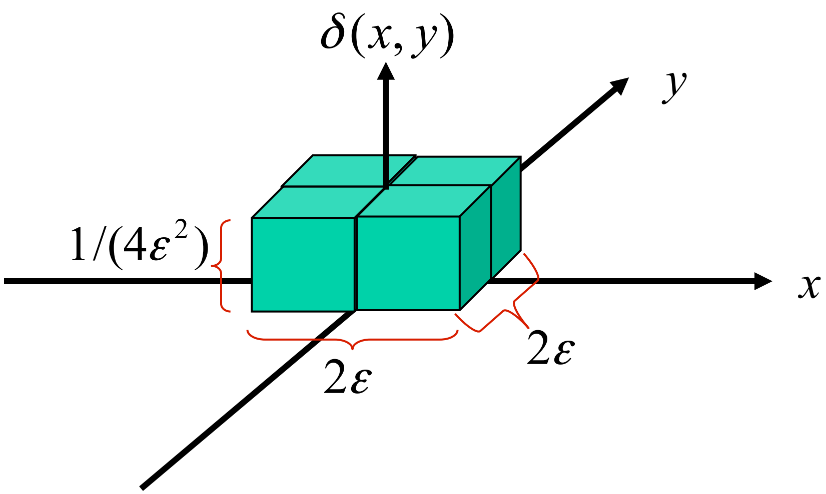
Basics: The point spread function

Relate h to an observable

- Given arbitrary $h(x,y)$, can we always find an $f(x,y)$ that produces $h(x,y)$ as output?

$$h(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)h(\xi,\eta)d\xi d\eta$$

- Apparently, the desired f must be zero at all points, except the origin.
- Further, we will let the integral of this function over any region including the origin be 1.
- We call the desired function the unit impulse and denote it as $\delta(x,y)$
- More formally, it is the limit as $\varepsilon \to 0$ of a series of square pulses of width 2ε in x and y and height $1/(4\varepsilon^2)$



Remark

- The total "mass" of $\delta(x,y)$ is

$$(2\varepsilon)(2\varepsilon)(1/4\varepsilon^2)=1$$

Basics: The point spread function

Relate h to an observable

- Given arbitrary $h(x,y)$, can we always find an $f(x,y)$ that produces $h(x,y)$ as output?

$$h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)h(\xi,\eta)d\xi d\eta$$

- Apparently, the desired f must be zero at all points, except the origin.
- Further, we will let the integral of this function over any region including the origin be 1.
- We call the desired function the unit impulse and denote it as $\delta(x,y)$
- More formally, it is the limit as $\varepsilon \to 0$ of a series of square pulses of width 2ε in x and y and height $1/(4\varepsilon^2)$

The sifting property

- We note that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)h(x,y)dxdy = h(0,0)$$

Basics: The point spread function

Definition

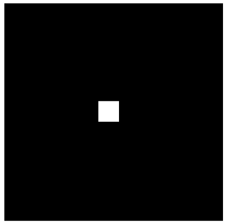
- Considered as an image, the unit impulse is black everywhere, except at the origin where there is a bright point of light.
- So $h(x,y)$ tells how the systems blurs or spreads out a point of light.

Basics: The point spread function

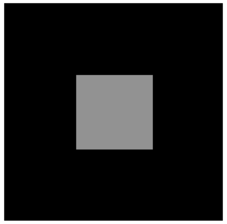
Definition

- Considered as an image, the unit impulse is black everywhere, except at the origin where there is a bright point of light.
- So $h(x,y)$ tells how the systems blurs or spreads out a point of light.
- As pictorial examples

$\delta(x, y)$ the unit impulse



$h(x,y)$ corresponding to our smoothing via local averaging example

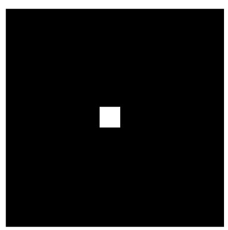


Basics: The point spread function

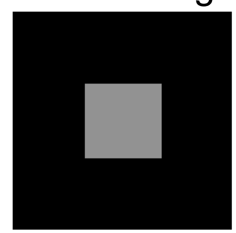
Definition

- Considered as an image, the unit impulse is black everywhere, except at the origin where there is a bright point of light.
- So $h(x,y)$ tells how the systems blurs or spreads out a point of light.
- As pictorial examples

$\delta(x, y)$ the unit impulse



$h(x,y)$ corresponding to our smoothing via local averaging example

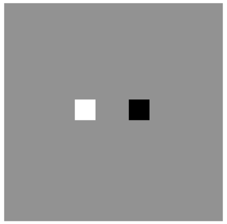


- We call h the point spread function.

Basics: The point spread function

Another example revisited

- We can take the 2D version of our 1, 0, -1 convolution example to have a point spread function, $h(x,y)$, with the appearance



- Apparently this h spreads out a single point of light as pair of opposite polarity points.

Basics: Overview

Linear, shift invariant systems

Convolution

The point-spread function

The modulation transfer function

Basics: The modulation transfer function (MTF)

Eigenfunctions

- An eigenfunction of a system is one that is simply multiplied by another factor in the output.

$$f(w) \longrightarrow \boxed{} \longrightarrow A(w)\,f(w)$$

- We think of this as analogous to the case of eigenvectors from linear algebra.

Basics: The modulation transfer function (MTF)

Eigenfunctions

- An eigenfunction of a system is one that is simply multiplied by another factor in the output.

- We think of this as analogous to the case of eigenvectors from linear algebra.

Remark

- Notation

$$e^{iwt} = \exp(iwt)$$

with the imaginary number

$$i = \sqrt{-1}$$

Basics: The modulation transfer function (MTF)

Eigenfunctions

- An eigenfunction of a system is one that is simply multiplied by another factor in the output.

$$f(w) \longrightarrow \boxed{} \longrightarrow A(w)\ \mathrm{f}(w)$$

- We think of this as analogous to the case of eigenvectors from linear algebra.
- For the case of 1D LSI systems we find that $\exp(iwt)$ is an eigenfunction of convolution.

$$\exp(iwt) \longrightarrow \boxed{} \longrightarrow A(w)\ \exp(iwt)$$

- Here $A(w)$ is the (possibly complex) factor by which the input signal is multiplied.
- So, from the input exponential we obtain another exponential; but, scaled and shifted in phase.

Basics: The modulation transfer function (MTF)

Eigenfunctions

- An eigenfunction of a system is one that is simply multiplied by another factor in the output.

$$f(w) \longrightarrow \boxed{\qquad} \longrightarrow A(w)\,\mathrm{f}(w)$$

- We think of this as analogous to the case of eigenvectors from linear algebra.
- For the case of 1D LSI systems we find that $\exp(iwt)$ is an eigenfunction of convolution.

$$\exp(iwt) \longrightarrow \boxed{\qquad} \longrightarrow A(w)\,\exp(iwt)$$

- Here $A(w)$ is the (possibly complex) factor by which the input signal is multiplied.
- So, from the input exponential we obtain another exponential; but, scaled and shifted in phase.

$$A(w) = S(w)\exp[i\varphi(w)]; S(w) \equiv scaling$$

$$\Rightarrow A(w)\exp(iwt) = S(w)\exp[i\varphi(w)]\exp(iwt)$$

$$= S(w)\exp\{i[wt + \varphi(w)]\}$$

Basics: The modulation transfer function (MTF)

Eigenfunctions

- An eigenfunction of a system is one that is simply multiplied by another factor in the output.

$$f(w) \longrightarrow \boxed{\qquad} \longrightarrow A(w)\ \mathrm{f}(w)$$

- We think of this as analogous to the case of eigenvectors from linear algebra.
- For the case of 1D LSI systems we find that $\exp(iwt)$ is an eigenfunction of convolution.

$$\exp(iwt) \longrightarrow \boxed{\qquad} \longrightarrow A(w)\ \exp(iwt)$$

- Here $A(w)$ is the (possibly complex) factor by which the input signal is multiplied.
- So, from the input exponential we obtain another exponential; but, scaled and shifted in phase.

Frequency

- We call w the frequency (or wave number) of the eigenfunction.
- In practice, we use real waveforms, like $\cos(wt)$ and $\sin(wt)$, with the relationship

$$\exp(iwt) = \cos(wt) + i \sin(wt)$$

  which is known as Euler's relation.
- The complex exponential is used in derivations simply because it provides a compact notation.

Basics: The modulation transfer function (MTF)

1D frequency

- We consider functions of the form

  $f(x) = A\cos(ux + d)$

  where

  A is the amplitude

  u is the (angular) frequency

  d is the phase constant.
- Notice that the function repeats its value when $ux + d$ increases by 2π.
- For example, when $d = 0$, the maxima and minima occur when $ux = k\pi$, for k an integer.
- The wavelength (period), λ, is defined by calculating when the argument to cos at $x + \lambda$ is 2π plus that at x

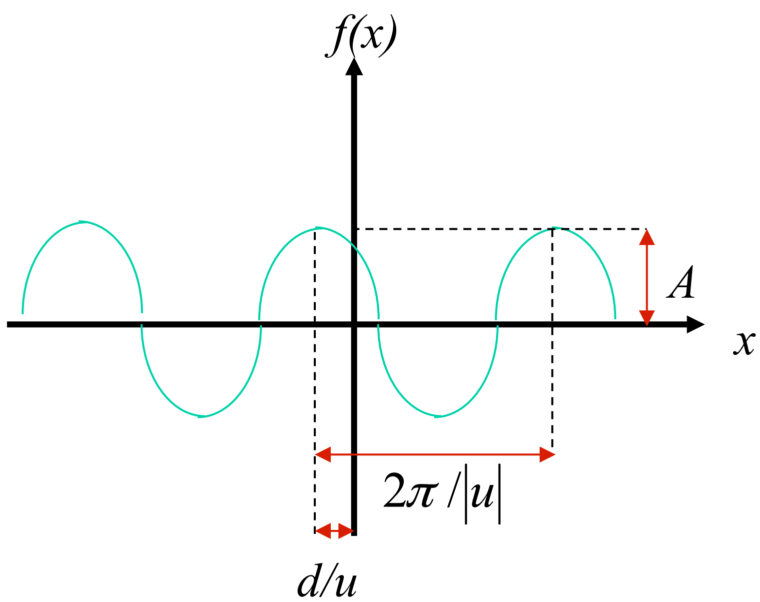
$$u(\cancel{x} + \lambda) + \cancel{d} = 2\pi + \cancel{u}x + \cancel{d}$$

$$\Rightarrow \lambda = 2\pi / |u|$$

- The shift in the peak (from 0) is

$$d/u$$

e.g., $\cos[u(-d/u) + d] = \cos[0] = 1$



Basics: The modulation transfer function (MTF)

2D Eigenfunctions

- For the case of 2D LSI systems we find that the input $f(x,y) = \exp(i(ux+vy))$ yields output

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{i[u(x-\xi)+v(y-\eta)]\} h(\xi,\eta)\, d\xi\, d\eta$$

Basics: The modulation transfer function (MTF)

2D Eigenfunctions

- For the case of 2D LSI systems we find that the input $f(x,y) = \exp(i(ux+vy))$ yields output

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{i[u(x-\xi)+v(y-\eta)]\}h(\xi,\eta)d\xi d\eta$$

* Recall

$$\exp(a+b) = \exp(a)\exp(b)$$

$$\Rightarrow \exp\{i[u(x-\xi)+v(y-\eta)]\} = \exp[i(ux+vy)]\exp[-i(u\xi+v\eta)]$$

Basics: The modulation transfer function (MTF)

2D Eigenfunctions

- For the case of 2D LSI systems we find that the input $f(x,y) = \exp(i(ux+vy))$ yields output

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{i[u(x-\xi)+v(y-\eta)]\}h(\xi,\eta)d\xi d\eta$$

or (*)

$$g(x,y) = \exp[i(ux+vy)]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-i(u\xi+v\eta)]h(\xi,\eta)d\xi d\eta$$

* Recall

$$\exp(a+b) = \exp(a)\exp(b)$$

$$\Rightarrow \exp\{i[u(x-\xi)+v(y-\eta)]\} = \exp[i(ux+vy)]\exp[-i(u\xi+v\eta)]$$

Basics: The modulation transfer function (MTF)

2D Eigenfunctions

- For the case of 2D LSI systems we find that the input $f(x,y) = \exp(i(ux+vy))$ yields output

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{i[u(x-\xi)+v(y-\eta)]\}h(\xi,\eta)d\xi d\eta$$

or

$$g(x,y) = \exp[i(ux+vy)]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-i(u\xi+v\eta)]h(\xi,\eta)d\xi d\eta$$

- The double integral on the right is a function of u and v only
- So, the output, $g(x,y)$, is just a scaled, possibly shifted, version of the input, $f(x,y)$

$$g(x,y) = f(x,y)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-i(u\xi+v\eta)]h(\xi,\eta)d\xi d\eta$$

Basics: The modulation transfer function (MTF)

2D Eigenfunctions

- For the case of 2D LSI systems we find that the input $f(x,y) = \exp(i(ux+vy))$ yields output

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{i[u(x-\xi)+v(y-\eta)]\}h(\xi,\eta)d\xi d\eta$$

or

$$g(x,y) = \exp[i(ux+vy)]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-i(u\xi+v\eta)]h(\xi,\eta)d\xi d\eta$$

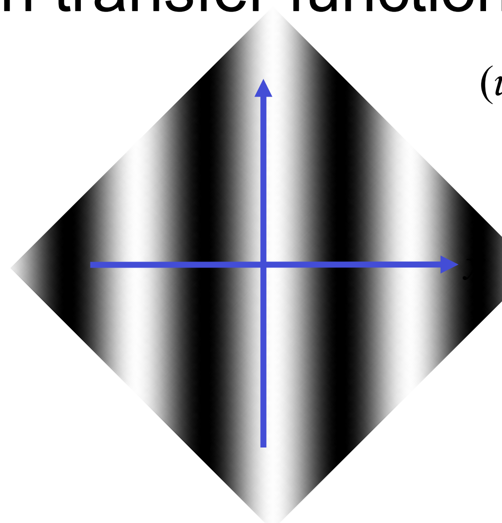
- The double integral on the right is a function of u and v only
- So, the output, $g(x,y)$, is just a scaled, possibly shifted, version of the input, $f(x,y)$

$$g(x,y) = f(x,y)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-i(u\xi+v\eta)]h(\xi,\eta)d\xi d\eta$$

- We conclude that $\exp[i(ux+vy)]$ is an eigenfunction in 2D.

$$\exp[i(ux+vy)] \longrightarrow \boxed{} \longrightarrow A(u,v)\exp[i(ux+vy)]$$

Basics: The modulation transfer function (MTF)

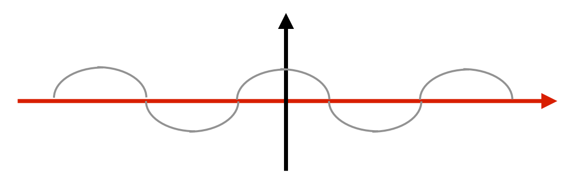


$(u,v)=(a,0)$

The maxima and minima of the cosinusoids lie along parallel equidistant lines $ux + vy = k\pi$ for k an integer.

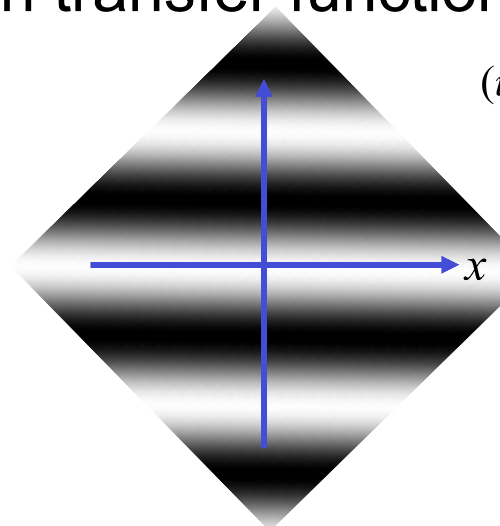
2D frequency

- For two spatial dimensions, we see that there are two corresponding frequency components, u and v.
- We refer to the uv-plane as the frequency domain.
- We refer to the xy-plane as the spatial domain.
- The real waveforms $cos(ux+vy)$ and $sin(ux+vy)$ correspond to waves in 2D.



Cross sections orthogonal to the ridges show a sinusoidal profile with wavelength $2\pi / \sqrt{u^2 + v^2}$

Basics: The modulation transfer function (MTF)

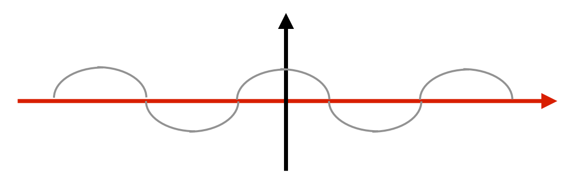


(u,v)=(0,a)

The maxima and minima of the cosinusoids lie along parallel equidistant lines $ux + vy = k\pi$ for k an integer.

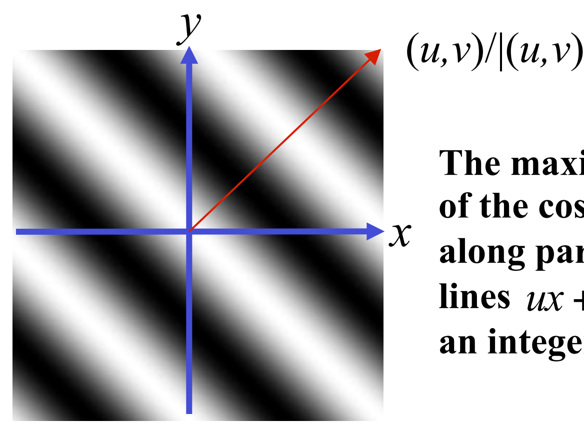
2D frequency

- For two spatial dimensions, we see that there are two corresponding frequency components, u and v.
- We refer to the uv-plane as the frequency domain.
- We refer to the xy-plane as the spatial domain.
- The real waveforms $cos(ux+vy)$ and $sin(ux+vy)$ correspond to waves in 2D.



Cross sections orthogonal to the ridges show a sinusoidal profile with wavelength $2\pi \big/ \sqrt{u^2 + v^2}$

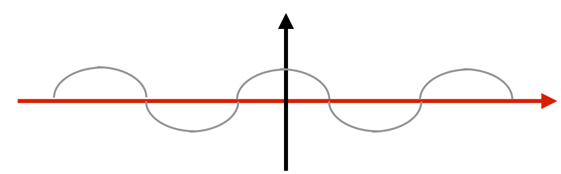
Basics: The modulation transfer function (MTF)



The maxima and minima of the cosinusoids lie along parallel equidistant lines $ux + vy = k\pi$ for k an integer.

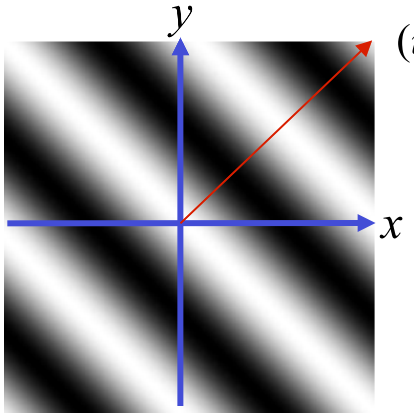
2D frequency

- For two spatial dimensions, we see that there are two corresponding frequency components, u and v.
- We refer to the uv-plane as the frequency domain.
- We refer to the xy-plane as the spatial domain.
- The real waveforms $cos(ux+vy)$ and $sin(ux+vy)$ correspond to waves in 2D.



Cross sections orthogonal to the ridges show a sinusoidal profile with wavelength $2\pi / \sqrt{u^2 + v^2}$

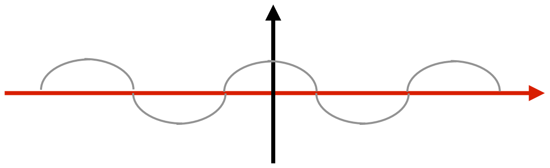
Basics: The modulation transfer function (MTF)



The maxima and minima of the cosinusoids lie along parallel equidistant lines $ux + vy = k\pi$ for k an integer.

2D frequency

- For two spatial dimensions, we see that there are two corresponding frequency components, u and v.
- We refer to the uv-plane as the frequency domain.
- We refer to the xy-plane as the spatial domain.
- The real waveforms $\cos(ux+vy)$ and $\sin(ux+vy)$ correspond to waves in 2D.



Cross sections orthogonal to the ridges show a sinusoidal profile with wavelength $2\pi / \sqrt{u^2 + v^2}$

Remark

- As sinusoids, these waves cannot occur on their own in an imaging system (with only positive values).
- By convention, we consider a constant additive offset.

Basics: The modulation transfer function (MTF)

Definition

- Let

$$H(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-i(u\xi + v\eta)]h(\xi,\eta)d\xi d\eta$$

Basics: The modulation transfer function (MTF)

Definition

- Let

$$H(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-i(u\xi + v\eta)]h(\xi,\eta)d\xi d\eta$$

- Then for the case treated so far

$$g(x,y) = \exp[i(ux+vy)]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-i(u\xi + v\eta)]h(\xi,\eta)d\xi d\eta$$

  with $f(x,y) = \exp(i(ux+vy))$, we have

$$g(x,y) = H(u,v)f(x,y)$$

- So, in some sense $H(u,v)$ characterizes the system for sinusoidal waveforms.
  - For each frequency, it tells the response of the system in amplitude and phase.
  - We refer to H as the modulation transfer function.

Basics: The modulation transfer function (MTF)

Definition

- Let

$$H(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-i(u\xi + v\eta)]h(\xi,\eta)d\xi d\eta$$

- Then for the case treated so far

$$g(x,y) = \exp[i(ux+vy)]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-i(u\xi + v\eta)]h(\xi,\eta)d\xi d\eta$$

  with $f(x,y) = \exp(i(ux+vy))$, we have

$$g(x,y) = H(u,v)f(x,y)$$

- So, in some sense $H(u,v)$ characterizes the system for sinusoidal waveforms.
  - For each frequency, it tells the response of the system in amplitude and phase.
  - We refer to H as the modulation transfer function.

Remark

- Apparently, if we knew how to decompose an arbitrary function, f, into a sum of sinusoidal waveforms, then the MTF could be used to characterize the effect of an LSI system operating on a function.
- It seems that this would be a simpler description (just multiplication!) than that offered by convolution with a PSF.

Basics: Recapitulation

Linear, shift invariant systems

Convolution

The point-spread function

The modulation transfer function

Outline

Basics

The Fourier transform

Local operators

Restoration and enhancement

The discrete case

Local scale and orientation

Fourier transform: Intuition

Consider

- A (periodic) signal with fundamental frequency 2 pi

$$f(x) = \sum_{k=-3}^{3} a_k \exp(ik2\pi x)$$

with

$$a_0 = 1$$
$$a_1 = a_{-1} = 1/4$$
$$a_2 = a_{-2} = 1/2$$
$$a_3 = a_{-3} = 1/3$$

Fourier transform: Intuition

Consider

- A (periodic) signal with fundamental frequency 2 pi

$$f(x) = \sum_{k=-3}^{3} a_k \exp(ik2\pi x)$$

with

$$a_0 = 1$$

$$a_1 = a_{-1} = 1/4$$

$$a_2 = a_{-2} = 1/2$$

$$a_3 = a_{-3} = 1/3$$

- A form like this "begs" to have common terms collected together

$$= 1 + \frac{1}{4}[\exp(i2\pi x) + \exp(-i2\pi x)]$$

$$+ \frac{1}{2}[\exp(i4\pi x) + \exp(-i4\pi x)]$$

$$+ \frac{1}{3}[\exp(i6\pi x) + \exp(-i6\pi x)]$$

Fourier transform: Intuition

Consider

- We further examine our expansion

$$= 1 + \frac{1}{4}[\exp(i2\pi x) + \exp(-i2\pi x)]$$

$$+ \frac{1}{2}[\exp(i4\pi x) + \exp(-i4\pi x)]$$

$$+ \frac{1}{3}[\exp(i6\pi x) + \exp(-i6\pi x)]$$

$$\exp(i2\pi x) + \exp(-i2\pi x)$$

$$= \cos(2\pi x) + i\sin(2\pi x) + \cos(-2\pi x) + i\sin(-2\pi x)$$

Recall: $\cos(x)=\cos(-x)$; $\sin(x)=-sin(-x)$

$$= 2\cos(2\pi x)$$

- and note that we can cancel terms inside the grouped exponents via Euler's relation to yield

$$= 1$$

$$+ \frac{1}{2}\cos 2\pi x$$

$$+ \cos 4\pi x$$

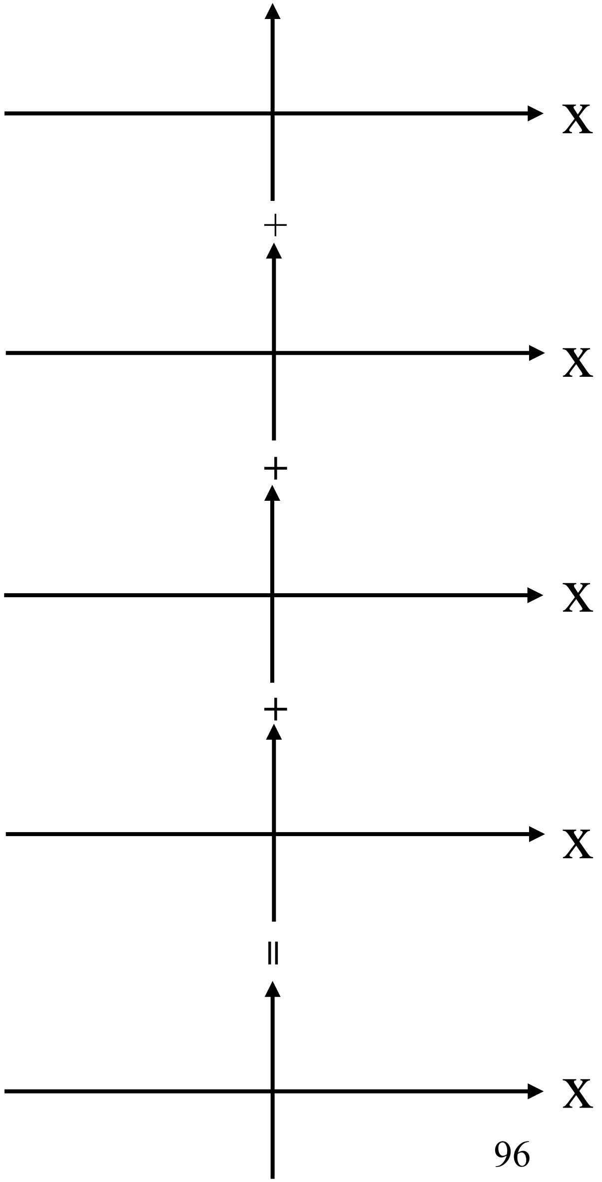
$$+ \frac{2}{3}\cos 6\pi x$$

Fourier transform: Intuition

Consider

- A graphical interpretation

$$
\begin{aligned}
&1 \\
&+\frac{1}{2}\cos 2\pi x \\
&+\cos 4\pi x \\
&+\frac{2}{3}\cos 6\pi x \\
&= f(x)
\end{aligned}
$$



Fourier transform: Intuition

Consider

- A graphical interpretation

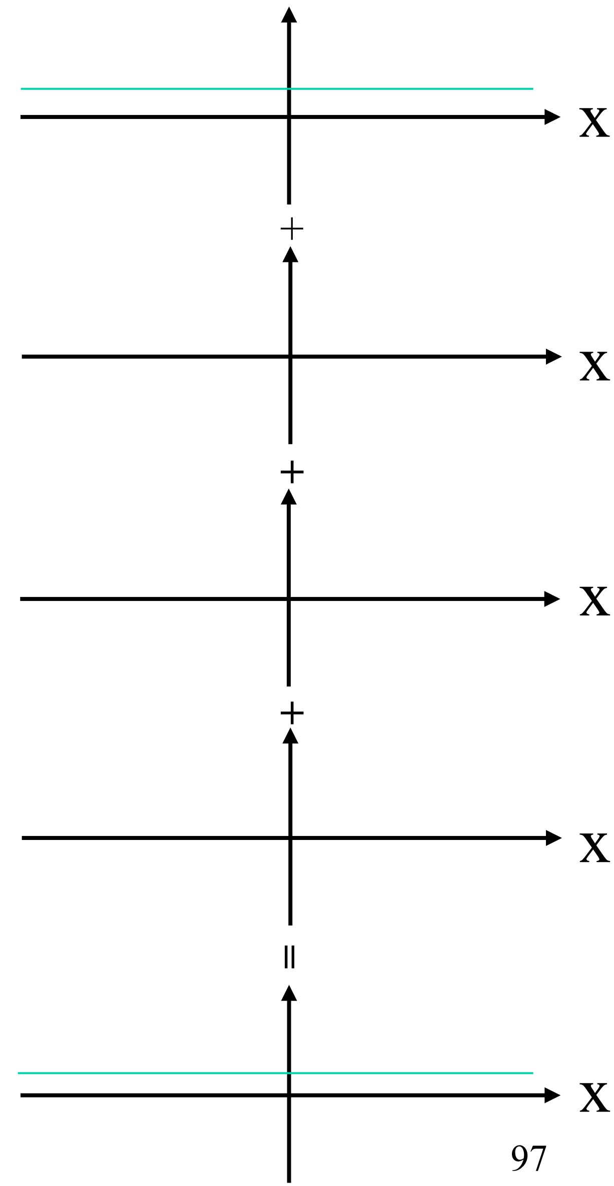
$$1$$

$$+\frac{1}{2}\cos 2\pi x$$

$$+\cos 4\pi x$$

$$+\frac{2}{3}\cos 6\pi x$$

$$= f(x)$$



Fourier transform: Intuition

Consider

- A graphical interpretation

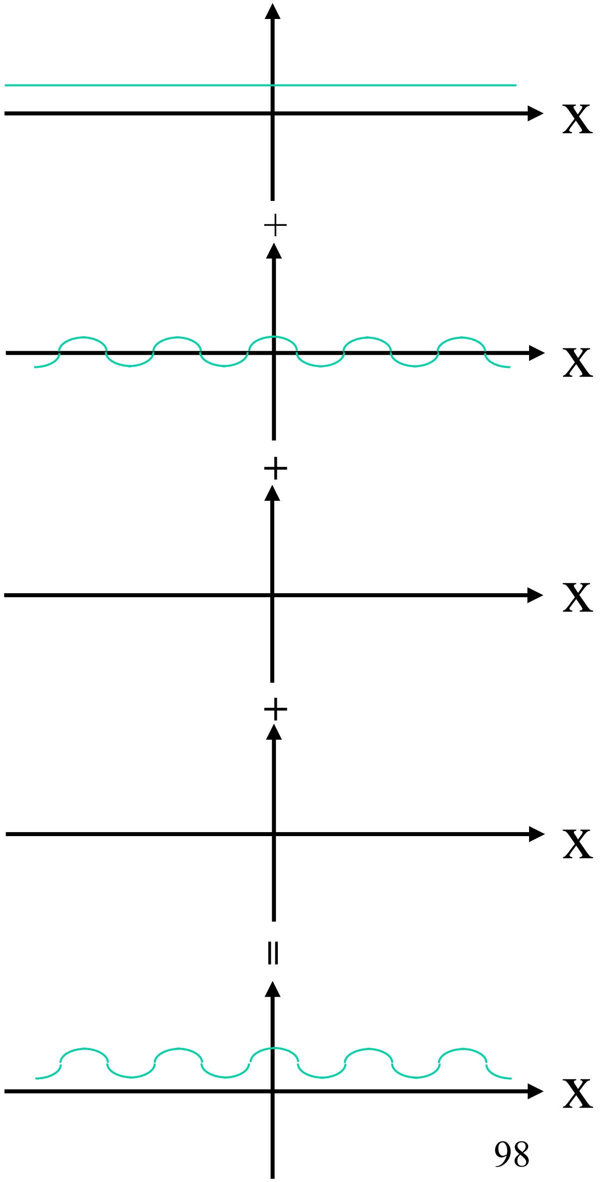
$$1$$

$$+\frac{1}{2}\cos 2\pi x$$

$$+\cos 4\pi x$$

$$+\frac{2}{3}\cos 6\pi x$$

$$=f(x)$$



Fourier transform: Intuition

Consider

- A graphical interpretation

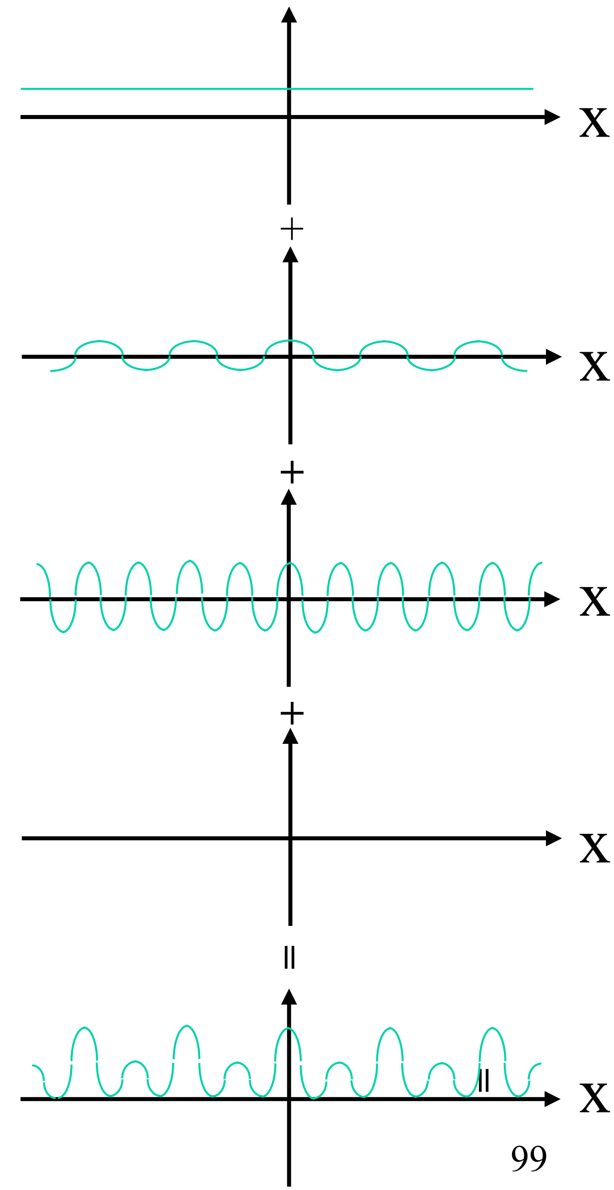
$$1$$

$$+\frac{1}{2}\cos 2\pi x$$

$$+\cos 4\pi x$$

$$+\frac{2}{3}\cos 6\pi x$$

$$= f(x)$$



Fourier transform: Intuition

Consider

- A graphical interpretation

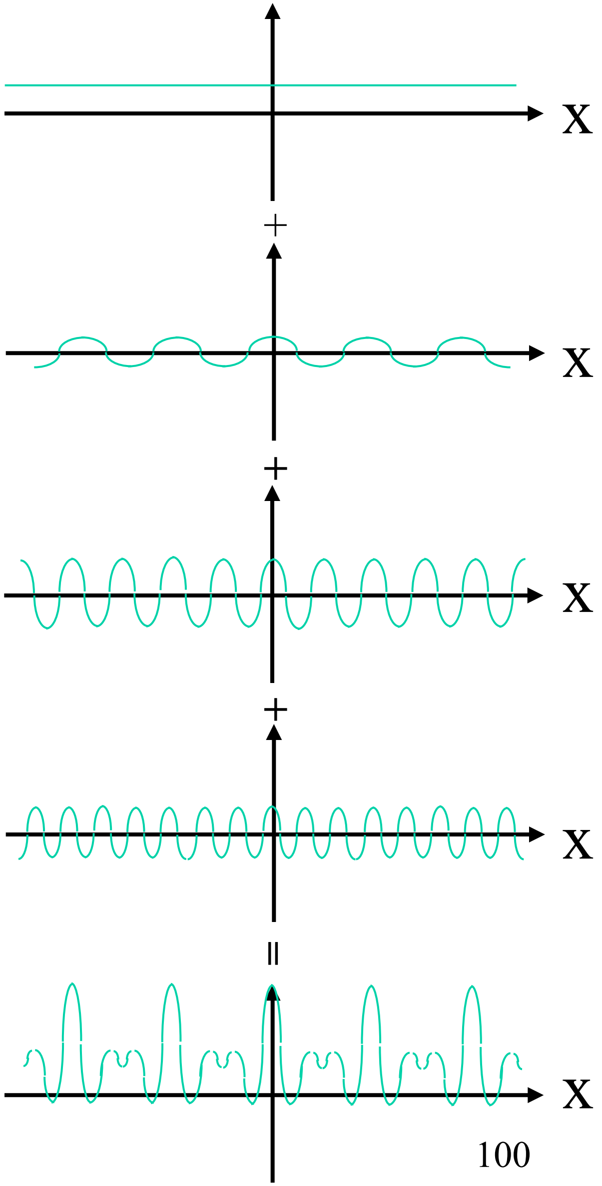
$$1$$

$$+\frac{1}{2}\cos 2\pi x$$

$$+\cos 4\pi x$$

$$+\frac{2}{3}\cos 6\pi x$$

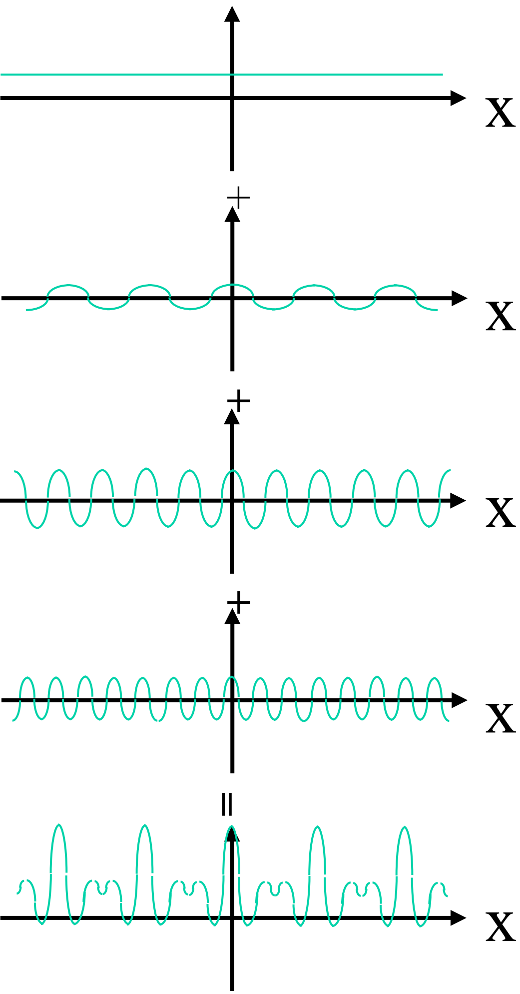
$$= f(x)$$



Observation

- Complicated signals can be represented as the sum of simple components.

Fourier transform: Intuition



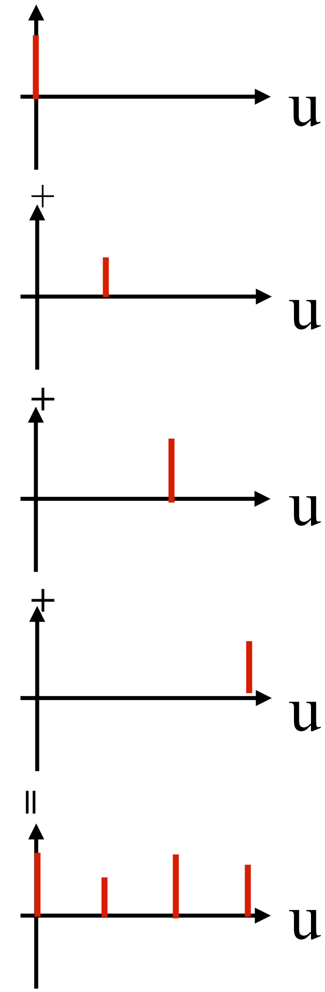
f(x)

$$1\cos 0\pi x \rightarrow 1\cos ux; u = 0$$

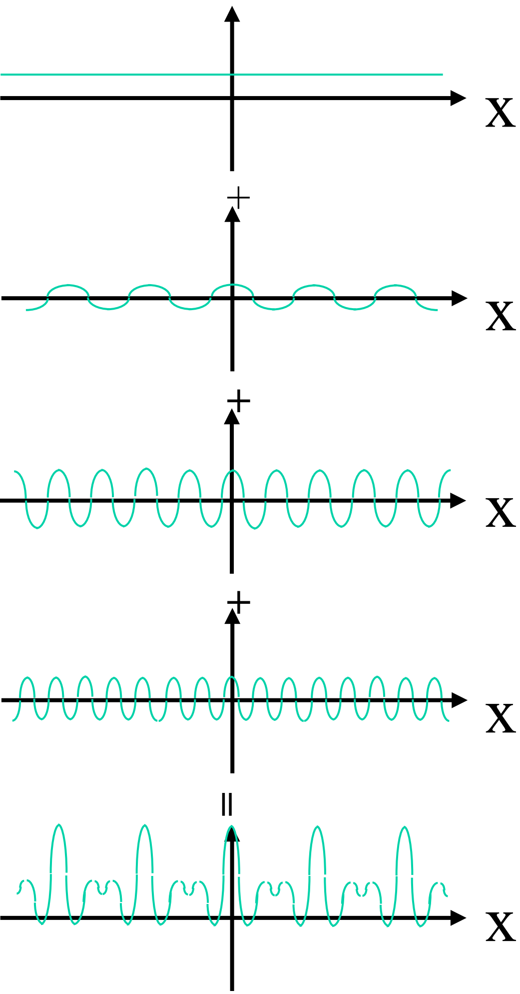
$$+\frac{1}{2}\cos 2\pi x \rightarrow +\frac{1}{2}\cos ux; u = 2\pi$$

$$+\cos 4\pi x \rightarrow +1\cos ux; u = 4\pi$$

$$+\frac{2}{3}\cos 6\pi x \rightarrow +\frac{2}{3}\cos ux; u = 6\pi$$



Fourier transform: Intuition



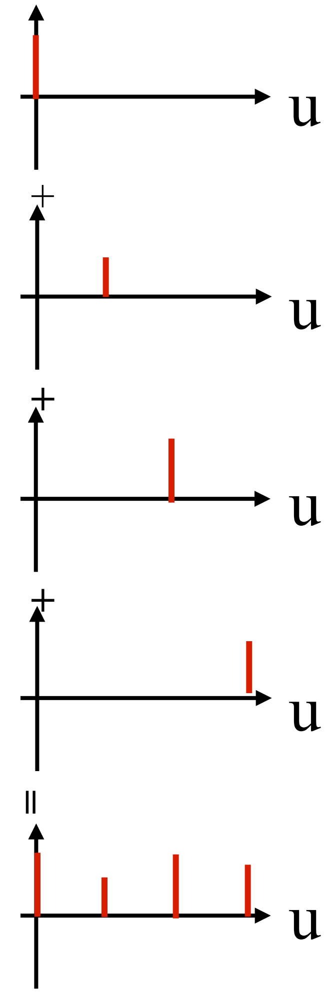
f(x)

$$1\cos 0\pi x \rightarrow 1\cos ux; u = 0$$

$$+\frac{1}{2}\cos 2\pi x \rightarrow +\frac{1}{2}\cos ux; u = 2\pi$$

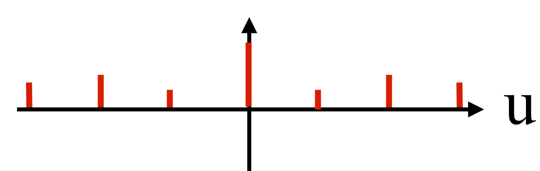
$$+\cos 4\pi x \rightarrow +1\cos ux; u = 4\pi$$

$$+\frac{2}{3}\cos 6\pi x \rightarrow +\frac{2}{3}\cos ux; u = 6\pi$$



Remark

- By symmetry, we may choose to represent this as



The Fourier transform: Filtering

Signal decomposition

- An input, $f(x,y)$, can be considered as the sum of an infinite number of sinusoidal waves.
- This is a convenient way to decompose the input as, provided the system MTF, $H(u,v)$, we know the system response to each component.

The Fourier transform: Filtering

Signal decomposition

- An input, $f(x,y)$, can be considered as the sum of an infinite number of sinusoidal waves.
- This is a convenient way to decompose the input as, provided the system MTF, $H(u,v)$, we know the system response to each component.
- Suppose we decompose as

$$f(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\exp[i(ux+vy)]dudv$$

The Fourier transform: Filtering

Signal decomposition

- An input, $f(x,y)$, can be considered as the sum of an infinite number of sinusoidal waves.
- This is a convenient way to decompose the input as, provided the system MTF, $H(u,v)$, we know the system response to each component.
- Suppose we decompose as

$$f(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\exp[i(ux+vy)]dudv$$

- By definition of convolution

$$g(x,y) = f(x,y) * h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)h(\xi,\eta)d\xi d\eta$$

- Previously, we showed that for $f(x,y) = \exp(i(ux+vy))$

$$g(x,y) = H(u,v)f(x,y)$$

The Fourier transform: Filtering

Signal decomposition

- An input, $f(x,y)$, can be considered as the sum of an infinite number of sinusoidal waves.
- This is a convenient way to decompose the input as, provided the system MTF, $H(u,v)$, we know the system response to each component.
- Suppose we decompose as

$$f(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\exp[i(ux+vy)]dudv$$

- By definition of convolution

$$g(x,y) = f(x,y) * h(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)h(\xi,\eta)d\xi d\eta$$

- Previously, we showed that for $f(x,y) = \exp(i(ux+vy))$

$$g(x,y) = H(u,v)f(x,y)$$

- So, for $f(x,y)$ an integral (infinite sum) of sinusoids, as we have hypothesized,

$$g(x,y) = f(x,y) * h(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(u,v)F(u,v)\exp[i(ux+vy)]dudv$$

The Fourier transform: Filtering

How to find *F(u,v)* given *f(x,y)*

- A useful definition is

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp[-i(ux+vy)]dxdy$$

provided the integral exists.

The Fourier transform: Filtering

How to find $F(u,v)$ given $f(x,y)$

- A useful definition is

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp[-i(ux+vy)]dxdy$$

provided the integral exists.

- To see that this makes sense, we substitute into the expression for $f(x,y)$

$$f(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\exp[i(ux+vy)]dudv$$

The Fourier transform: Filtering

How to find ***F(u,v)*** **given** ***f(x,y)***

- A useful definition is

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp[-i(ux+vy)]dxdy$$

provided the integral exists.

- To see that this makes sense, we substitute into the expression for $f(x,y)$

$$f(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\exp[i(ux+vy)]dudv$$

and use of a change of variables (so that we avoid x, y standing for two different things)

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a,b)\exp[-i(ua+vb)]dadb$$

to obtain

$$f(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a,b)\exp[-i(ua+vb)dadb\right]\exp[i(ux+vy)]dudv$$

The Fourier transform: Filtering

How to find $F(u,v)$ given $f(x,y)$

- A useful definition is

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp[-i(ux+vy)]dxdy$$

provided the integral exists.

- To see that this makes sense, we substitute into the expression for $f(x,y)$

$$f(x,y)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\exp[i(ux+vy)]dudv$$

and use of a change of variables (so that we avoid x, y standing for two different things)

$$F(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a,b)\exp[-i(ua+vb)]dadb$$

to obtain

$$f(x,y)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a,b)\exp[-i(ua+vb)dadb\right]\exp[i(ux+vy)]dudv$$

or

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a,b)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{i[(u(x-a)+v(y-b)]\}dudv\right]dadb$$

The Fourier transform: Filtering

How to find *F(u,v)* given *f(x,y)*

- We want to understand, in terms of F, the integral

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a,b)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{i[(u(x-a)+v(y-b)]\}dudv\right]dadb$$

The Fourier transform: Filtering

How to find $F(u,v)$ given $f(x,y)$

- We want to understand, in terms of F, the integral

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a,b)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{i[(u(x-a)+v(y-b)]\}dudv\right]dadb$$

- It can be shown that the inner integral can be taken to evaluate to $4\pi^2\delta(x-a,y-b)$

The Fourier transform: Filtering

How to find $F(u,v)$ given $f(x,y)$

- We want to understand, in terms of F, the integral

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a,b)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{i[(u(x-a)+v(y-b)]\}dudv\right]dadb$$

- It can be shown that the inner integral can be taken to evaluate to $4\pi^2\delta(x-a,y-b)$
- This allows us to write

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a,b)\delta(x-a,y-b)dadb$$

The Fourier transform: Filtering

How to find *F(u,v)* given *f(x,y)*

- We want to understand, in terms of F, the integral

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a,b)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{i[(u(x-a)+v(y-b)]\}dudv\right]dadb$$

- It can be shown that the inner integral can be taken to evaluate to $4\pi^2\delta(x-a,y-b)$
- This allows us to write

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a,b)\delta(x-a,y-b)dadb = \delta(x,y) * f(x,y) = f(x,y)$$

The Fourier transform: Filtering

How to find *F(u,v)* given *f(x,y)*

- We want to understand, in terms of F, the integral

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a,b)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{i[(u(x-a)+v(y-b)]\}dudv\right]dadb$$

- It can be shown that the inner integral can be taken to evaluate to $4\pi^2\delta(x-a,y-b)$
- This allows us to write

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a,b)\delta(x-a,y-b)dadb = \delta(x,y) * f(x,y) = f(x,y)$$

- We have now come full cycle and see that our earlier choice for *F(u,v)* as

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp[-i(ux+vy)]dxdy$$

yields

$$f(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\exp[i(ux+vy)]dudv$$

The Fourier transform: Filtering

How to find *F(u,v)* given *f(x,y)*

- We want to understand, in terms of F, the integral

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a,b)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{i[(u(x-a)+v(y-b)]\}dudv\right]dadb$$

- It can be shown that the inner integral can be taken to evaluate to $4\pi^2\delta(x-a,y-b)$
- This allows us to write

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(a,b)\delta(x-a,y-b)dadb = \delta(x,y) * f(x,y) = f(x,y)$$

- We have now come full cycle and see that our earlier choice for $F(u,v)$ as

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp[-i(ux+vy)]dxdy$$

yields

$$f(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\exp[i(ux+vy)]dudv$$

- We call $F(u,v)$ the Fourier transform of $f(x,y)$.

The Fourier transform: Filtering

Definition

- Sometimes, we will find it convenient to consider the (squared) magnitude of the Fourier transform

$$\left|F(u,v)\right|^2$$

- We call this function the power spectrum of f.
- For small $\delta u, \delta v$

$$\left|F(u,v)\right|^2 \delta u \delta v$$

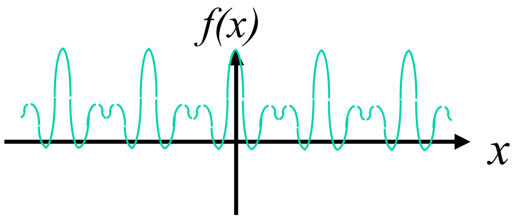
gives the power in the rectangular region of the frequency domain lying between $u, u+\delta u$ and $v, v+\delta v$

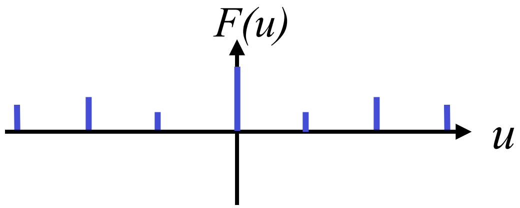
- We take this as a measure of the magnitude or "energy" of the signal in that frequency interval, independent of phase information.

The Fourier transform: Filtering

A 2D example

- Recall the 1D example



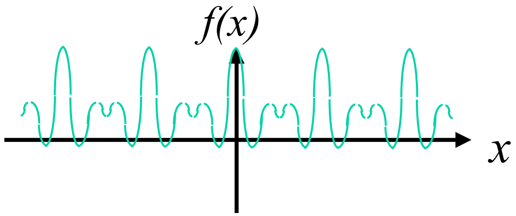


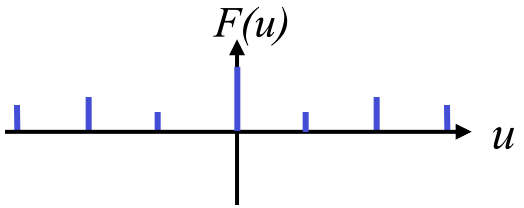
for the cosine component

The Fourier transform: Filtering

A 2D example

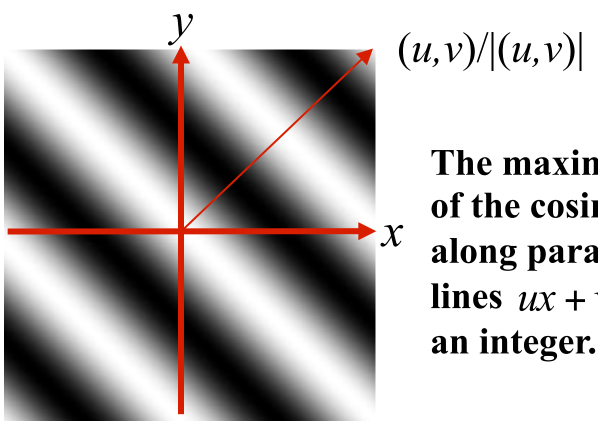
- Recall the 1D example





for the cosine component

- And the interpretation of 2D spatial frequency

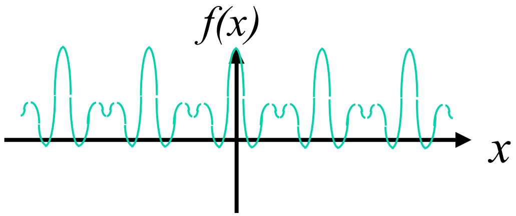


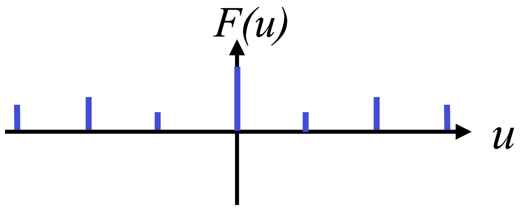
The maxima and minima of the cosinusoids lie along parallel equidistant lines $ux + vy = k\pi$ **for** $\boldsymbol{k}$ **an integer.**

The Fourier transform: Filtering

A 2D example

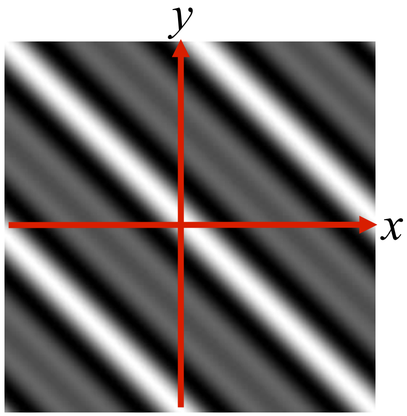
- Recall the 1D example





for the cosine component

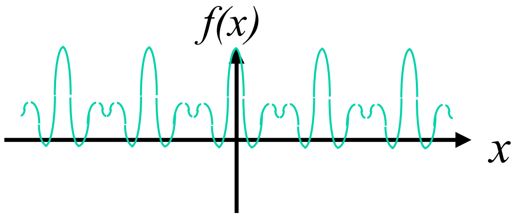
- Then a 2D analogue could be

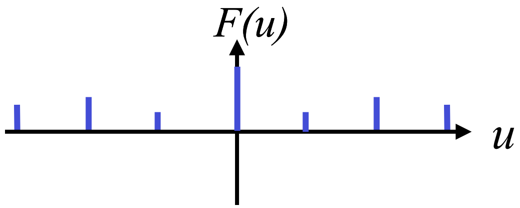


The Fourier transform: Filtering

A 2D example

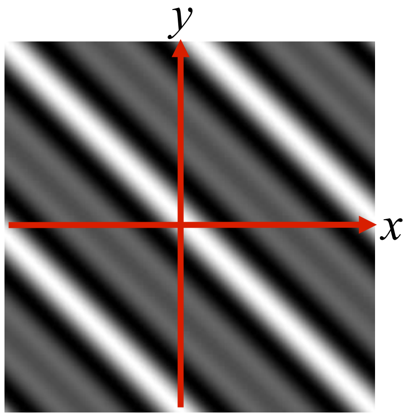
- Recall the 1D example

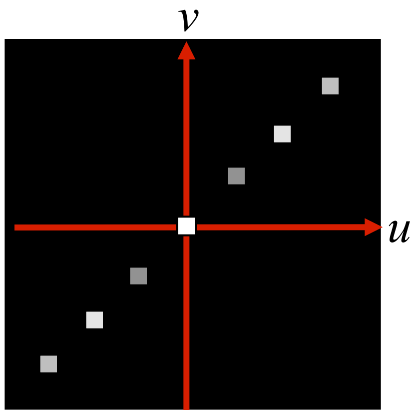




for the cosine component

- Then a 2D analogue could be





The Fourier transform: Filtering

Transforming convolution into multiplication

- Let $g = f*h$, then the Fourier transform $G(u,v)$ of $g(x,y)$ is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)h(\xi,\eta)d\xi d\eta\right]\exp[-i(ux+vy)]dxdy$$

The Fourier transform: Filtering

Transforming convolution into multiplication

- Let $g = f*h$, then the Fourier transform $G(u,v)$ of $g(x,y)$ is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)h(\xi,\eta)d\xi d\eta\right]\exp[-i(ux+vy)]dxdy$$

- Changing the order of integration, we can write

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)\exp[-i(ux+vy)]dxdy\right]h(\xi,\eta)d\xi d\eta$$

The Fourier transform: Filtering

Transforming convolution into multiplication

- Let $g = f*h$, then the Fourier transform $G(u,v)$ of $g(x,y)$ is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)h(\xi,\eta)d\xi d\eta\right]\exp[-i(ux+vy)]dxdy$$

- Changing the order of integration, we can write

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)\exp[-i(ux+vy)]dxdy\right]h(\xi,\eta)d\xi d\eta$$

- We recognize that the terms in the inner integral are close to that of a Fourier transform, which can be made exact via the substitution

$$\alpha = x-\xi, \quad x = \alpha+\xi, \quad dx = d\alpha$$
$$\beta = y-\eta, \quad y = \beta+\eta, \quad dy = d\beta$$

The Fourier transform: Filtering

Transforming convolution into multiplication

- Let $g = f*h$, then the Fourier transform $G(u,v)$ of $g(x,y)$ is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)h(\xi,\eta)d\xi d\eta\right]\exp[-i(ux+vy)]dxdy$$

- Changing the order of integration, we can write

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)\exp[-i(ux+vy)]dxdy\right]h(\xi,\eta)d\xi d\eta$$

- We recognize that the terms in the inner integral are close to that of a Fourier transform, which can be made exact via the substitution

$$\alpha = x-\xi, \quad x = \alpha+\xi, \quad dx = d\alpha$$
$$\beta = y-\eta, \quad y = \beta+\eta, \quad dy = d\beta$$

which yields

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\exp\{-i[u(\alpha+\xi)+v(\beta+\eta)]\}d\alpha d\beta\right]h(\xi,\eta)d\xi d\eta$$

The Fourier transform: Filtering

Transforming convolution in multiplication

- We have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\exp\{-i[u(\alpha+\xi)+v(\beta+\eta)]\}d\alpha d\beta\right]h(\xi,\eta)d\xi d\eta$$

The Fourier transform: Filtering

Transforming convolution in multiplication

- We have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\exp\{-i[u(\alpha+\xi)+v(\beta+\eta)]\}d\alpha d\beta\right]h(\xi,\eta)d\xi d\eta$$

- Our immediate interest is in alpha and beta; so, noting that $\exp(m+n) = \exp(m)\exp(n)$, we substitute and rearrange to get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\exp[-i(u\alpha+v\beta)]\exp[-i(u\xi+v\eta)]d\alpha d\beta\right]h(\xi,\eta)d\xi d\eta$$

The Fourier transform: Filtering

Transforming convolution in multiplication

- We have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\exp\{-i[u(\alpha+\xi)+v(\beta+\eta)]\}d\alpha d\beta\right]h(\xi,\eta)d\xi d\eta$$

- Our immediate interest is in alpha and beta; so, noting that $\exp(m+n) = \exp(m)\exp(n)$, we substitute and rearrange to get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\exp[-i(u\alpha+v\beta)]\exp[-i(u\xi+v\eta)]d\alpha d\beta\right]h(\xi,\eta)d\xi d\eta$$

- Now, recognizing the exact form for the Fourier transform of f we write

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\exp[-i(u\xi+v\eta)]h(\xi,\eta)d\xi d\eta$$

The Fourier transform: Filtering

Transforming convolution in multiplication

- We have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\exp\{-i[u(\alpha+\xi)+v(\beta+\eta)]\}d\alpha d\beta\right]h(\xi,\eta)d\xi d\eta$$

- Our immediate interest is in alpha and beta; so, noting that $\exp(m+n) = \exp(m)\exp(n)$, we substitute and rearrange to get

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\alpha,\beta)\exp[-i(u\alpha+v\beta)]\exp[-i(u\xi+v\eta)]d\alpha d\beta\right]h(\xi,\eta)d\xi d\eta$$

- Now, recognizing the exact form for the Fourier transform of f we write

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\exp[-i(u\xi+v\eta)]h(\xi,\eta)d\xi d\eta$$

- Similarly, recognizing the Fourier transform of h we write

$$F(u,v)H(u,v)=G(u,v)$$

- We conclude that the convolution of two functions in the spatial domain corresponds to taking the product of the two transformed functions in the Fourier domain.
- Notably, we see that H is simply the MTF of our linear system.

The Fourier transform: Filtering

Recapitulation

- We have seen that denoting the Fourier transform of the system output, $g(x,y)=f(x,y)*h(x,y)$, as $G(u,v)$ we can write

$$G(u,v) = H(u,v)F(u,v)$$

  where F is the Fourier transform of f and H, the MTF, is the Fourier transform of h.
- Notice the simplicity of the previous expression as compared to

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)h(\xi,\eta)d\xi d\eta$$

- More generally, we have seen that
  – Convolution in the spatial domain corresponds to multiplication in the frequency domain
  – The converse is true as well

The Fourier transform: Filtering

Recapitulation

- We have seen that denoting the Fourier transform of the system output, $g(x,y)=f(x,y)*h(x,y)$, as $G(u,v)$ we can write

$$G(u,v) = H(u,v)F(u,v)$$

  where F is the Fourier transform of f and H, the MTF, is the Fourier transform of h.
- Notice the simplicity of the previous expression as compared to

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-\xi, y-\eta)h(\xi,\eta)d\xi d\eta$$

- More generally, we have seen that
  - Convolution in the spatial domain corresponds to multiplication in the frequency domain
  - The converse is true as well

Remarks

- Once again we see that the MTF specifies how a system attenuates or amplifies each component $F(u,v)$ of the input.
- More generally we note that an LSI system acts as a filter that alters the amplitude and phase of the frequency components of its input, but that is all.

The Fourier transform: Recapitulation

$$F(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp[-i(ux+vy)]dxdy$$

Fourier transform

Spatial domain

- $f(x,y)$
- Convolution
- Multiplication
- Point spread function

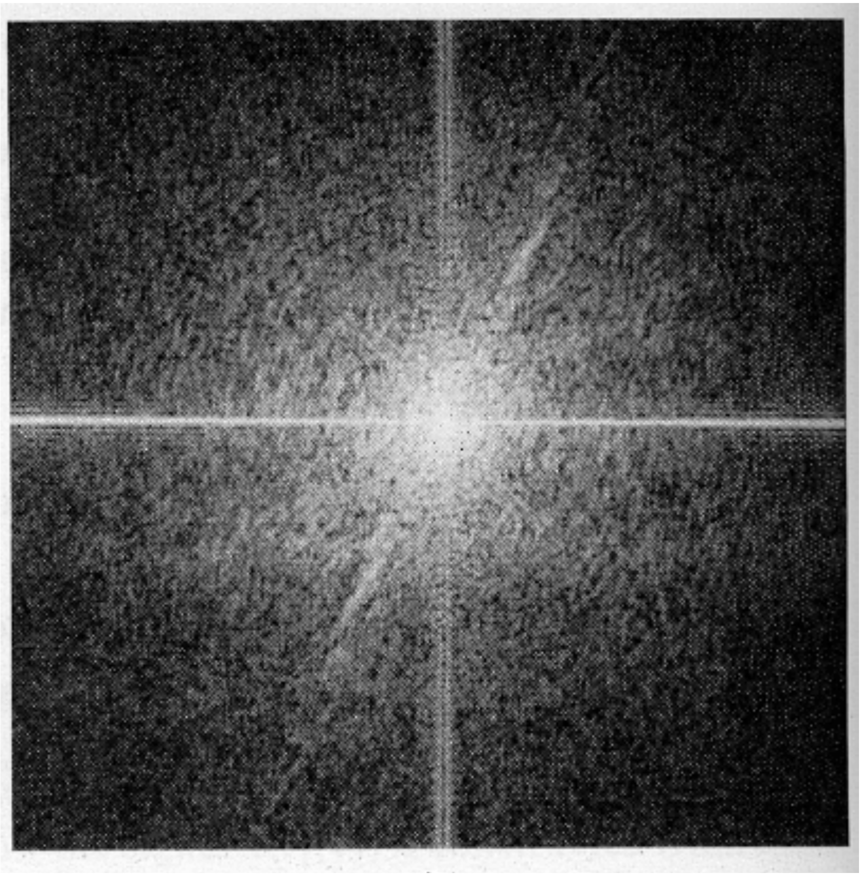
Frequency domain

- $F(u,v)$
- Multiplication
- Convolution
- Modulation transfer function

Inverse Fourier transform

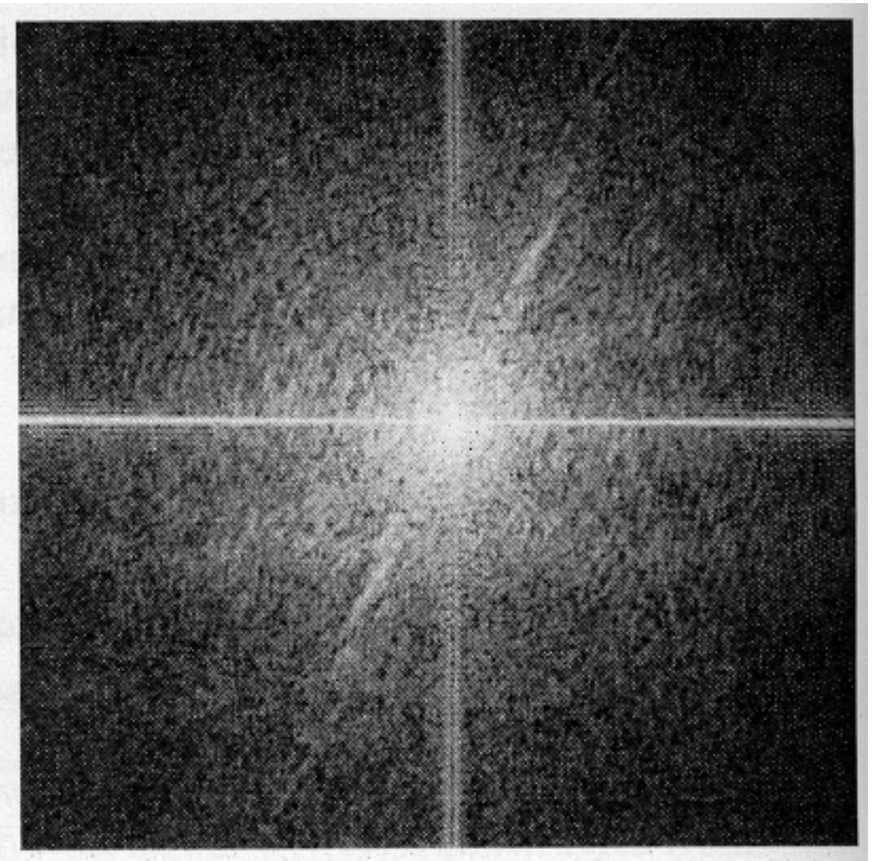
$$f(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\exp[i(ux+vy)]dudv$$

The Fourier transform: Recapitulation



Fourier power spectrum

The Fourier transform: Recapitulation



Source image (J. Fourier)

Fourier power spectrum

Outline

Basics

The Fourier transform

Local operators

Restoration and enhancement

The discrete case

Local scale and orientation

Local operators: Partial derivatives and convolution

Motivation

- We shall use differentiation to accentuate edges in images.
- Therefore, it will be useful to know how the Fourier transform of the derived images is related to the Fourier transform of the original image.
- In particular, if $F(u,v)$ is the Fourier transform of $f(x,y)$, then what are the Fourier transforms of $\partial f / \partial x$ and $\partial f / \partial y$?

Local operators: Partial derivatives and convolution

Motivation

- We shall use differentiation to accentuate edges in images.
- Therefore, it will be useful to know how the Fourier transform of the derived images is related to the Fourier transform of the original image.
- In particular, if $F(u,v)$ is the Fourier transform of $f(x,y)$, then what are the Fourier transforms of $\partial f / \partial x$ and $\partial f / \partial y$?

Derivation

- Consider the transform

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial f}{\partial x} \exp[-i(ux + vy)]dxdy$$

Local operators: Partial derivatives and convolution

Motivation

- We shall use differentiation to accentuate edges in images.
- Therefore, it will be useful to know how the Fourier transform of the derived images is related to the Fourier transform of the original image.
- In particular, if $F(u,v)$ is the Fourier transform of $f(x,y)$, then what are the Fourier transforms of $\partial f/\partial x$ and $\partial f/\partial y$?

Derivation

- Consider the transform

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial f}{\partial x}\exp[-i(ux+vy)]dxdy$$

- We break this up as

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\frac{\partial f}{\partial x}\exp(-iux)dx\right]\exp(-ivy)dy$$

Local operators: Partial derivatives and convolution

Motivation

- We shall use differentiation to accentuate edges in images.
- Therefore, it will be useful to know how the Fourier transform of the derived images is related to the Fourier transform of the original image.
- In particular, if $F(u,v)$ is the Fourier transform of $f(x,y)$, then what are the Fourier transforms of $\partial f / \partial x$ and $\partial f / \partial y$?

Derivation

- Consider the transform

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial f}{\partial x}\exp[-i(ux+vy)]dxdy$$

- We break this up as

$$\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\frac{\partial f}{\partial x}\exp(-iux)dx\right]\exp(-ivy)dy$$

- The inner integral yields to integration by parts

Integration by parts substitution

$$\int udv = uv - \int vdu$$

$$\begin{aligned} u &= \exp[-iux] \\ dv &= \frac{\partial f}{\partial x}dx \\ v &= f \\ du &= -iu\exp[-iux]dx \end{aligned}$$

$$\int_{-\infty}^{\infty}\frac{\partial f}{\partial x}\exp(-iux)dx = \left[f(x,y)\exp(-iux)\right]_{-\infty}^{\infty} + iu\int_{-\infty}^{\infty} f(x,y)\exp(-iux)dx$$

Local operators: Partial derivatives and convolution

Motivation

- We shall use differentiation to accentuate edges in images.
- Therefore, it will be useful to know how the Fourier transform of the derived images is related to the Fourier transform of the original image.
- In particular, if $F(u,v)$ is the Fourier transform of $f(x,y)$, then what are the Fourier transforms of $\partial f / \partial x$ and $\partial f / \partial y$?

Derivation

- Consider the transform

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial f}{\partial x} \exp[-i(ux+vy)]dxdy$$

- We break this up as

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \frac{\partial f}{\partial x} \exp(-iux)dx\right] \exp(-ivy)dy$$

- The inner integral yields to integration by parts

$$\int_{-\infty}^{\infty} \frac{\partial f}{\partial x} \exp(-iux)dx = \left[f(x,y)\exp(-iux)\right]_{-\infty}^{\infty} + iu\int_{-\infty}^{\infty} f(x,y)\exp(-iux)dx$$

- Assuming that $f(x,y) \rightarrow 0$ as $x \rightarrow \pm\infty$, we can write

$$iu\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp[-i(ux+vy)]dxdy$$

Local operators: Partial derivatives and convolution

Motivation

- We shall use differentiation to accentuate edges in images.
- Therefore, it will be useful to know how the Fourier transform of the derived images is related to the Fourier transform of the original image.
- In particular, if $F(u,v)$ is the Fourier transform of $f(x,y)$, then what are the Fourier transforms of $\partial f / \partial x$ and $\partial f / \partial y$?

Derivation

- Consider the transform

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial f}{\partial x} \exp[-i(ux+vy)]dxdy$$

- We break this up as

$$\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} \frac{\partial f}{\partial x} \exp(-iux)dx\right] \exp(-ivy)dy$$

- The inner integral yields to integration by parts

$$\int_{-\infty}^{\infty} \frac{\partial f}{\partial x} \exp(-iux)dx = \left[f(x,y)\exp(-iux)\right]_{-\infty}^{\infty} + iu\int_{-\infty}^{\infty} f(x,y)\exp(-iux)dx$$

- Assuming that $f(x,y) \to 0$ as $x \to \pm\infty$, we can write

$$iu\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\exp[-i(ux+vy)]dxdy = iuF(u,v)$$

Local operators: Partial derivatives and convolution

Conclusion

- We have found that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial f}{\partial x}\exp[-i(ux+vy)]dxdy = iuF(u,v)$$

- Further, a similar derivation will yield

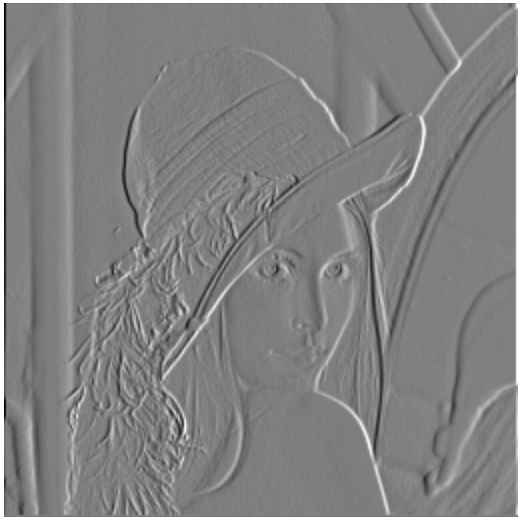
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial f}{\partial y}\exp[-i(ux+vy)]dxdy = ivF(u,v)$$

- We conclude that differentiation accentuates high frequency content at the expense of low frequency content.
- Indeed, 0 frequency content (constant off-set) is lost completely.

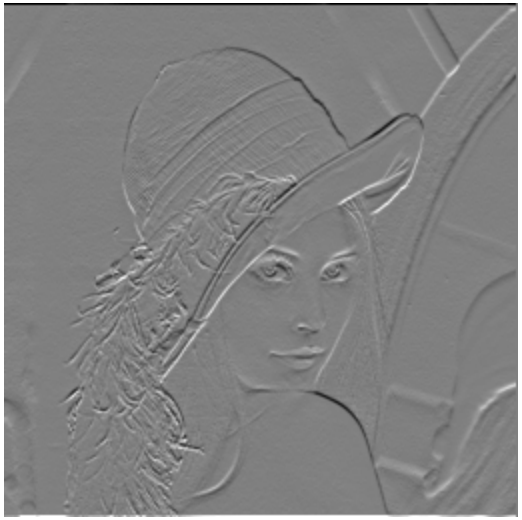
Local operators: Partial derivatives and convolution

Original image:

Differentiated images:



d/*dx*



d/*dy*

Local operators: Partial derivatives and convolution

A closer look

- We wonder, why does taking derivatives in the spatial domain correspond to multiplication in the frequency domain?
- But then we recall that differentiation is a LSI operation; so, it must be a convolution in the spatial domain and multiplication in the frequency domain.
- This brings another question: What is the function with which we convolve in the spatial domain to yield (partial) differentiation?

Local operators: Partial derivatives and convolution

A closer look

- We wonder, why does taking derivatives in the spatial domain correspond to multiplication in the frequency domain?
- But then we recall that differentiation is a LSI operation; so, it must be a convolution in the spatial domain and multiplication in the frequency domain.
- This brings another question: What is the function with which we convolve in the spatial domain to yield (partial) differentiation?
- We have seen that the Fourier transform of $\partial f / \partial x$ is $iuF(u,v)$ and the MTF of $\partial / \partial x$ is iu.
- The point spread function corresponding to the MTF is found by taking the inverse transform of iu

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} iu \exp[i(ux+vy)]dudv$$

Local operators: Partial derivatives and convolution

A closer look

- We wonder, why does taking derivatives in the spatial domain correspond to multiplication in the frequency domain?
- But then we recall that differentiation is a LSI operation; so, it must be a convolution in the spatial domain and multiplication in the frequency domain.
- This brings another question: What is the function with which we convolve in the spatial domain to yield (partial) differentiation?
- We have seen that the Fourier transform of $\partial f / \partial x$ is $iuF(u,v)$ and the MTF of $\partial / \partial x$ is iu.
- The point spread function corresponding to the MTF is found by taking the inverse transform of iu

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} iu \exp[i(ux+vy)]dudv$$

- Recalling our earlier convention that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[i(ux+vy)]dudv = 4\pi^2\delta(x,y)$$

Local operators: Partial derivatives and convolution

A closer look

- We wonder, why does taking derivatives in the spatial domain correspond to multiplication in the frequency domain?
- But then we recall that differentiation is a LSI operation; so, it must be a convolution in the spatial domain and multiplication in the frequency domain.
- This brings another question: What is the function with which we convolve in the spatial domain to yield (partial) differentiation?
- We have seen that the Fourier transform of $\partial f / \partial x$ is $iuF(u,v)$ and the MTF of $\partial / \partial x$ is iu.
- The point spread function corresponding to the MTF is found by taking the inverse transform of iu

$$\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} iu \exp[i(ux+vy)]dudv$$

- Recalling our earlier convention that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[i(ux+vy)]dudv = 4\pi^2\delta(x,y)$$

- and the fact that multiplication of the transform with iu corresponds to differentiation WRT x, we evaluate the integral of concern as

$$\frac{\partial}{\partial x}\delta(x,y)$$

Local operators: Partial derivatives and convolution

Another conclusion

- We have found that the point spread function corresponding to partial differentiation (in the x direction) is

$$\frac{\partial}{\partial x}\delta(x,y)$$

- But, how do we interpret this creature?

Local operators: Partial derivatives and convolution

Another conclusion

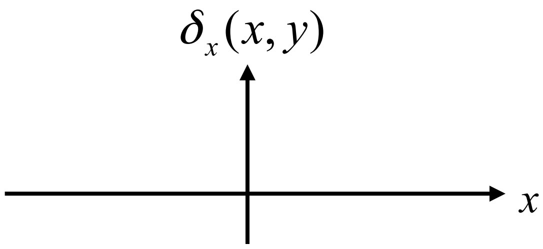
- We have found that the point spread function corresponding to partial differentiation (in the x direction) is

$$\frac{\partial}{\partial x}\delta(x,y)$$

- But, how do we interpret this creature?
- The delta function already is a so called “generalized function”.
- So, we expect to need special care for the definition of its derivative.
- We think of it as the limit of the sequence

$$\delta_x(x,y)=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}[\delta(x+\varepsilon,y)-\delta(x-\varepsilon,y)]$$

where we have two closely spaced impulses of opposite polarity.



Local operators: Partial derivatives and convolution

Another conclusion

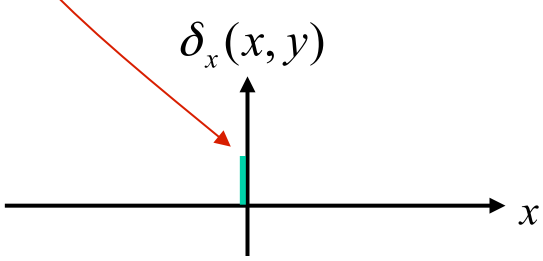
- We have found that the point spread function corresponding to partial differentiation (in the x direction) is

$$\frac{\partial}{\partial x}\delta(x,y)$$

- But, how do we interpret this creature?
- The delta function already is a so called "generalized function".
- So, we expect to need special care for the definition of its derivative.
- We think of it as the limit of the sequence

$$\delta_x(x,y) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon}[\delta(x+\varepsilon,y) - \delta(x-\varepsilon,y)]$$

where we have two closely spaced impulses of opposite polarity.



Local operators: Partial derivatives and convolution

Another conclusion

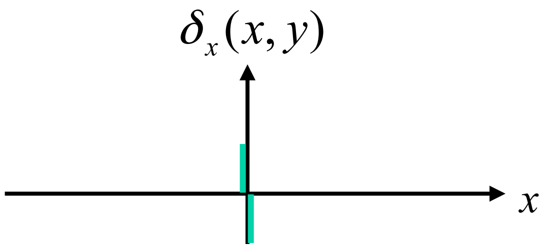
- We have found that the point spread function corresponding to partial differentiation (in the x direction) is

$$\frac{\partial}{\partial x}\delta(x,y)$$

- But, how do we interpret this creature?
- The delta function already is a so called “generalized function”.
- So, we expect to need special care for the definition of its derivative.
- We think of it as the limit of the sequence

$$\delta_x(x,y) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon}[\delta(x+\varepsilon,y) - \delta(x-\varepsilon,y)]$$

where we have two closely spaced impulses of opposite polarity.



Local operators: Partial derivatives and convolution

Another conclusion

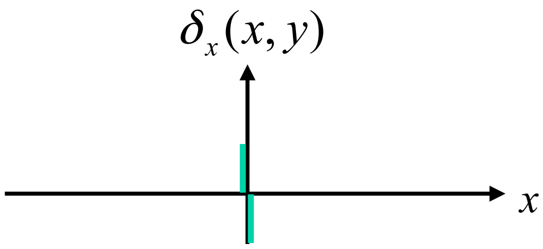
- We have found that the point spread function corresponding to partial differentiation (in the x direction) is

$$\frac{\partial}{\partial x}\delta(x,y)$$

- But, how do we interpret this creature?
- The delta function already is a so called “generalized function”.
- So, we expect to need special care for the definition of its derivative.
- We think of it as the limit of the sequence

$$\delta_x(x,y)=\lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}[\delta(x+\varepsilon,y)-\delta(x-\varepsilon,y)]$$

where we have two closely spaced impulses of opposite polarity.
- We call the result the doublet and denote it as $\delta_x(x,y)$



Local operators: Partial derivatives and convolution

Another conclusion

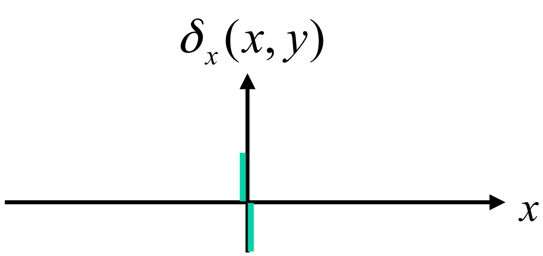
- We have found that the point spread function corresponding to partial differentiation (in the x direction) is

$$\frac{\partial}{\partial x}\delta(x,y)$$

- But, how do we interpret this creature?
- The delta function already is a so called "generalized function".
- So, we expect to need special care for the definition of its derivative.
- We think of it as the limit of the sequence

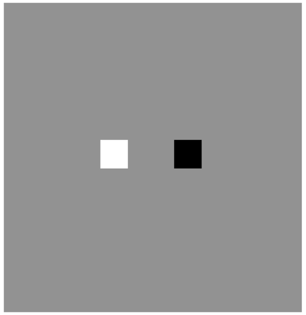
$$\delta_x(x,y) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon}[\delta(x+\varepsilon,y) - \delta(x-\varepsilon,y)]$$

where we have two closely spaced impulses of opposite polarity.
- We call the result the doublet and denote it as $\delta_x(x,y)$



Remark

- Recall our earlier example 2D PSF…
- which we now recognize as an approx. of $\delta_x(x,y)$



Local operators: Partial derivatives and convolution

Another conclusion

- We have found that the point spread function corresponding to partial differentiation (in the x direction) is

$$\frac{\partial}{\partial x}\delta(x,y)$$

- But, how do we interpret this creature?
- The delta function already is a so called "generalized function".
- So, we expect to need special care for the definition of its derivative.
- We think of it as the limit of the sequence

$$\delta_x(x,y) = \lim_{\varepsilon\to 0}\frac{1}{2\varepsilon}[\delta(x+\varepsilon,y)-\delta(x-\varepsilon,y)]$$

where we have two closely spaced impulses of opposite polarity.

- We call the result the doublet and denote it as $\delta_x(x,y)$
- Note that this definition corresponds to the usual definition of a partial derivative as the limit of a difference

$$f(x,y)*\delta_x(x,y) = \lim_{\varepsilon\to 0}\frac{f(x+\varepsilon,y)-f(x-\varepsilon,y)}{2\varepsilon} = \frac{\partial f}{\partial x}$$

- So, all turns out well in the end.

Outline

Basics

The Fourier transform

Local operators

Restoration and enhancement

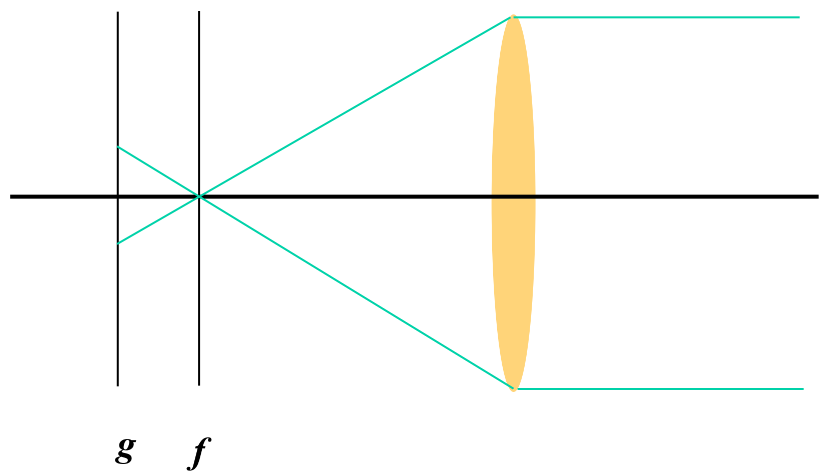
The discrete case

Local scale and orientation

Restoration and enhancement: Blur

Bad blur

- In a real-world imaging system we find that light rays that ideally would be focused at a point are (slightly) spread out.



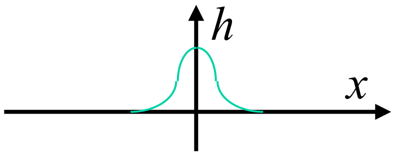
- Here, we think of g as a defocused version of f.

Restoration and enhancement: Blur

Bad blur

- In a real-world imaging system we find that light rays that ideally would be focused at a point are (slightly) spread out.
- Such blurring can sometimes be modeled via a Gaussian point spread function

$$h(x,y) = \frac{1}{2\pi\sigma^2}\exp\left[-\frac{1}{2}\left(\frac{x^2+y^2}{\sigma^2}\right)\right]$$



with sigma, the standard deviation that gives the spread of the Gaussian.

- This point spread function is rotationally symmetric as it depends only on $x^2 + y^2$, not x and y individually.
- To understand what is going on, let's compute the Fourier transform of this point spread function (i.e., the system MTF).

Restoration and enhancement: Blur

Fourier transform of the Gaussian

- We want to evaluate

$$H(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} \exp\left[-\frac{1}{2}\left(\frac{x^2+y^2}{\sigma^2}\right)\right] \exp[-i(ux+vy)]dxdy$$

Restoration and enhancement: Blur

Fourier transform of the Gaussian

- We want to evaluate

$$H(u,v)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi\sigma^2}\exp\left[-\frac{1}{2}\left(\frac{x^2+y^2}{\sigma^2}\right)\right]\exp[-i(ux+vy)]dxdy$$

- Begin be noticing that the 2D Gaussian can be separated into the product of two functions, so

$$=\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right]\exp(-iux)dx\,\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\left(\frac{y}{\sigma}\right)^2\right]\exp(-ivy)dy$$

Restoration and enhancement: Blur

Fourier transform of the Gaussian

- We want to evaluate

$$H(u,v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} \exp\left[-\frac{1}{2}\left(\frac{x^2+y^2}{\sigma^2}\right)\right] \exp[-i(ux+vy)]dxdy$$

- Begin be noticing that the 2D Gaussian can be separated into the product of two functions, so

$$= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right] \exp(-iux)dx \, \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} \exp\left[-\frac{1}{2}\left(\frac{y}{\sigma}\right)^2\right] \exp(-ivy)dy$$

- Let $a = 1/(2\sigma^2)$ so that the first integral on the RHS can be written more compactly as

$$\int_{-\infty}^{\infty} \exp(-ax^2)\exp(-iux)dx$$

or

$$\int_{-\infty}^{\infty} \exp[-a(x^2 + iux/a)]dx$$

Restoration and enhancement: Blur

Fourier transform of the Gaussian

- A form like

$$\int_{-\infty}^{\infty} \exp[-a(x^2 + iux/a)]dx$$

is best attacked by completing the square.

Restoration and enhancement: Blur

Fourier transform of the Gaussian

- A form like

$$\int_{-\infty}^{\infty} \exp[-a(x^2 + iux/a)]dx$$

  is best attacked by completing the square.
- Multiply the integrand by $\exp(u^2/4a)\exp(-u^2/4a)$ to yield

$$\int_{-\infty}^{\infty} \exp[-a(x^2 + iux/a)]\exp(u^2/4a)\exp(-u^2/4a)dx$$

  or

$$\int_{-\infty}^{\infty} \exp\{-a[x^2 + (iux/a) - (u^2/4a^2)]\}\exp(-u^2/4a)dx$$

Restoration and enhancement: Blur

Fourier transform of the Gaussian

- A form like

$$\int_{-\infty}^{\infty} \exp[-a(x^2 + iux/a)]dx$$

  is best attacked by completing the square.
- Multiply the integrand by $\exp(u^2/4a)\exp(-u^2/4a)$ to yield

$$\int_{-\infty}^{\infty} \exp[-a(x^2 + iux/a)]\exp(u^2/4a)\exp(-u^2/4a)dx$$

  or

$$\int_{-\infty}^{\infty} \exp\{-a[x^2 + (iux/a) - (u^2/4a^2)]\}\exp(-u^2/4a)dx$$

- We rearrange this more compactly as

$$\exp(-u^2/4a)\int_{-\infty}^{\infty} \exp\{-[\sqrt{a}(x + iu/2a)]^2\}dx$$

Restoration and enhancement: Blur

Fourier transform of the Gaussian

- We have

$$\exp(-u^2/4a)\int_{-\infty}^{\infty}\exp\{-[\sqrt{a}\,(x+iu/2a)]^2\}dx$$

Restoration and enhancement: Blur

Fourier transform of the Gaussian

- We have

$$\exp(-u^2/4a)\int_{-\infty}^{\infty}\exp\{-[\sqrt{a}(x+iu/2a)]^2\}dx$$

- It would be nice to make the exponent simpler, so we introduce a change of variable

$$t=\sqrt{a}\left(x+\frac{iu}{2a}\right)$$

$$dt=\sqrt{a}dx$$

and we have

$$\frac{1}{\sqrt{a}}\exp(-u^2/4a)\int_{-\infty}^{\infty}\exp(-t^2)dt$$

Restoration and enhancement: Blur

Fourier transform of the Gaussian

- Looking at

$$\frac{1}{\sqrt{a}}\exp(-u^2/4a)\int_{-\infty}^{\infty}\exp(-t^2)dt$$

  brings to mind that

- Fact:

$$\int_{-\infty}^{\infty}\exp(-\tau^2)d\tau = \sqrt{\pi}$$

Restoration and enhancement: Blur

Fourier transform of the Gaussian

- Looking at

$$\frac{1}{\sqrt{a}}\exp(-u^2/4a)\int_{-\infty}^{\infty}\exp(-t^2)dt$$

  brings to mind that
- Fact:

$$\int_{-\infty}^{\infty}\exp(-\tau^2)d\tau = \sqrt{\pi}$$

- So, the integral of concern evaluates to

$$\sqrt{\pi/a}\,\exp(-u^2/4a)$$

Restoration and enhancement: Blur

Fourier transform of the Gaussian

- Looking at

$$\frac{1}{\sqrt{a}} \exp(-u^2 / 4a) \int_{-\infty}^{\infty} \exp(-t^2) dt$$

  brings to mind that

- Fact:

$$\int_{-\infty}^{\infty} \exp(-\tau^2) d\tau = \sqrt{\pi}$$

- So, the integral of concern evaluates to

$$\sqrt{\pi / a} \exp(-u^2 / 4a)$$

  or, (recalling that $a = 1/(2\sigma^2)$)

$$\sqrt{2\pi}\sigma \exp\left[-\frac{1}{2}(u\sigma)^2\right]$$

Restoration and enhancement: Blur

Fourier transform of the Gaussian

- So, we have found that the first integral in

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right]\exp(-iux)dx\,\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\left(\frac{y}{\sigma}\right)^2\right]\exp(-ivy)dy$$

evaluates to

$$\sqrt{2\pi}\sigma\exp\left[-\frac{1}{2}(u\sigma)^2\right]$$

Restoration and enhancement: Blur

Fourier transform of the Gaussian

- So, we have found that the first integral in

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2\right]\exp(-iux)dx\,\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2}\left(\frac{y}{\sigma}\right)^2\right]\exp(-ivy)dy$$

evaluates to

$$\sqrt{2\pi}\sigma\exp\left[-\frac{1}{2}(u\sigma)^2\right]$$

- Not surprisingly, the second integral evaluates similarly.
- Overall, we have

$$\left\{\frac{1}{\sqrt{2\pi}\sigma}\sqrt{2\pi}\sigma\exp\left[-\frac{1}{2}(u\sigma)^2\right]\right\}\left\{\frac{1}{\sqrt{2\pi}\sigma}\sqrt{2\pi}\sigma\exp\left[-\frac{1}{2}(v\sigma)^2\right]\right\}$$

i.e.,

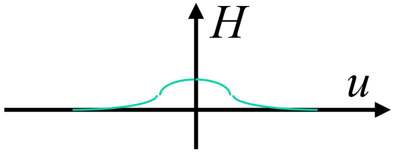
$$\exp\left[-\frac{1}{2}\left(u^2+v^2\right)\sigma^2\right]=H(u,v)$$

Restoration and enhancement: Blur

Gaussian MTF

- W e have found that the MTF of a Gaussian point-spread function is itself a Gaussian

$$H(u,v) = \exp\left[-\frac{1}{2}\left(u^2 + v^2\right)\sigma^2\right]$$



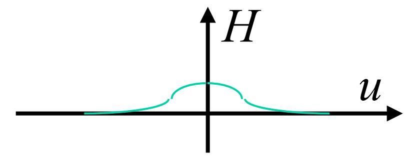
.

Restoration and enhancement: Blur

Gaussian MTF

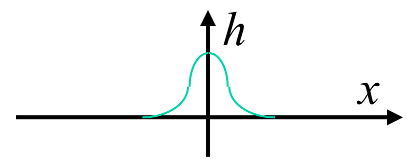
- W e have found that the MTF of a Gaussian point-spread function is itself a Gaussian

$$H(u,v) = \exp\left[-\frac{1}{2}\left(u^2 + v^2\right)\sigma^2\right]$$



- While $H(u,v)$ has a Gaussian shape, it has a spread that is the inverse of the spread of the point spread function

$$h(x,y) = \frac{1}{2\pi\sigma^2}\exp\left[-\frac{1}{2}\left(\frac{x^2+y^2}{\sigma^2}\right)\right]$$



- This is an example of the more general inverse relationship between scale changes in the spatial and frequency domains.
- Lower frequencies pass relatively unattenuated.
- Higher frequencies are reduced in amplitude.

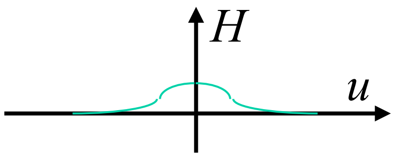
.

Restoration and enhancement: Blur

Gaussian MTF

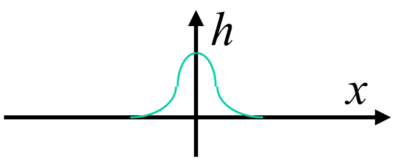
- W e have found that the MTF of a Gaussian point-spread function is itself a Gaussian

$$H(u,v) = \exp\left[-\frac{1}{2}\left(u^2+v^2\right)\sigma^2\right]$$



- While $H(u,v)$ has a Gaussian shape, it has a spread that is the inverse of the spread of the point spread function

$$h(x,y) = \frac{1}{2\pi\sigma^2}\exp\left[-\frac{1}{2}\left(\frac{x^2+y^2}{\sigma^2}\right)\right]$$



- This is an example of the more general inverse relationship between scale changes in the spatial and frequency domains.
- Lower frequencies pass relatively unattenuated.
- Higher frequencies are reduced in amplitude.

Good blur

- A number of image noise sources yield selective corruption of the high spatial frequencies.
- These effects can be ameliorated via application of (convolution with) a Gaussian point spread function.

Restoration and enhancement: Blur

Noise corrupted image | **Gaussian blurred image**

Restoration and enhancement: Beyond blur

The general case

- In a certain precise sense, application of a Gaussian blur point spread function for the amelioration of image corruption (noise) is optimal only if the noise is Gaussian.
- More generally, we would seek to derive and apply a point spread function whose MTF $H(u,v)$ is the inverse of that of the corruption, $N(u,v)$.

Restoration and enhancement: Beyond blur

The general case

- In a certain precise sense, application of a Gaussian blur point spread function for the amelioration of image corruption (noise) is optimal only if the noise is Gaussian.
- More generally, we would seek to derive and apply a point spread function whose MTF $H(u,v)$ is the inverse of that of the corruption, $N(u,v)$.
- This brings us to the topic of optimal filtering and the work of Wiener, Kolmogorov, and others…a path we will not follow any further.

Outline

Basics

The Fourier transform

Local operators

Restoration and enhancement

The discrete case

Local scale and orientation

The discrete case: Discrete image sampling

The forward transform

- When the image is digitized, the irradiance is known only at a discrete set of locations.

The discrete case: Discrete image sampling

The forward transform

- When the image is digitized, the irradiance is known only at a discrete set of locations.
- We think of the results as defined by a discrete grid of impulses

$$f(x,y) = wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl} \delta(x - kw, y - lh)$$

where w and h are the horizontal and vertical sampling intervals, respectively.

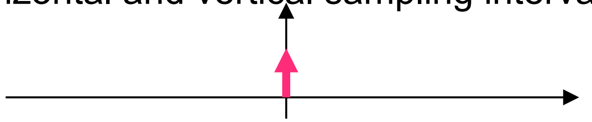
The discrete case: Discrete image sampling

The forward transform

- When the image is digitized, the irradiance is known only at a discrete set of locations.
- We think of the results as defined by a discrete grid of impulses

$$f(x,y) = wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl}\,\delta(x-kw, y-lh)$$

where w and h are the horizontal and vertical sampling intervals, respectively.



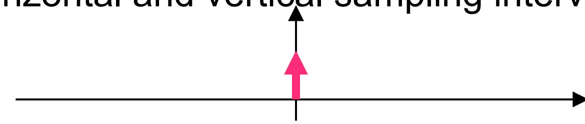
The discrete case: Discrete image sampling

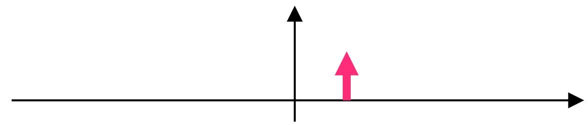
The forward transform

- When the image is digitized, the irradiance is known only at a discrete set of locations.
- We think of the results as defined by a discrete grid of impulses

$$f(x,y) = wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl} \delta(x - kw, y - lh)$$

where w and h are the horizontal and vertical sampling intervals, respectively.





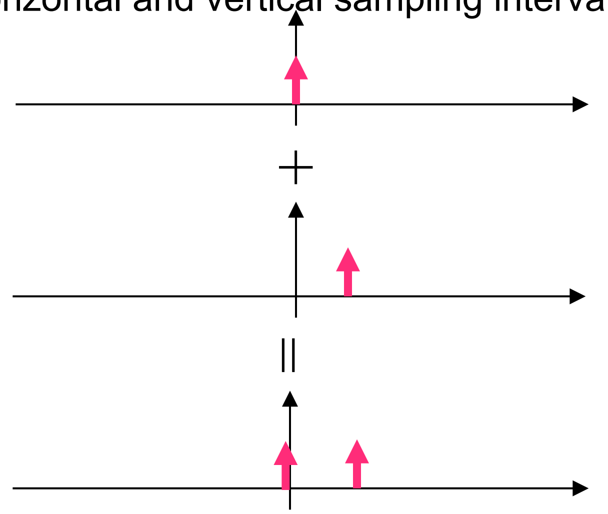
The discrete case: Discrete image sampling

The forward transform

- When the image is digitized, the irradiance is known only at a discrete set of locations.
- We think of the results as defined by a discrete grid of impulses

$$f(x, y) = wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl} \delta(x - kw, y - lh)$$

where w and h are the horizontal and vertical sampling intervals, respectively.



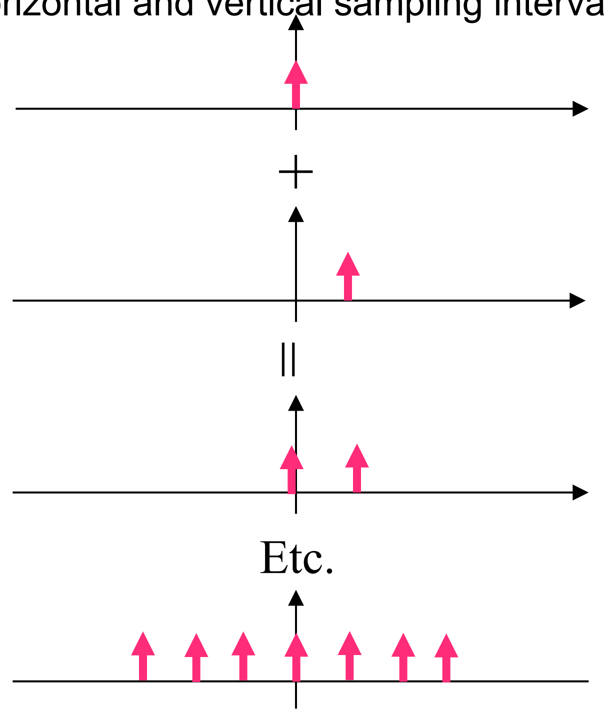
The discrete case: Discrete image sampling

The forward transform

- When the image is digitized, the irradiance is known only at a discrete set of locations.
- We think of the results as defined by a discrete grid of impulses

$$f(x, y) = wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl} \delta(x - kw, y - lh)$$

where w and h are the horizontal and vertical sampling intervals, respectively.



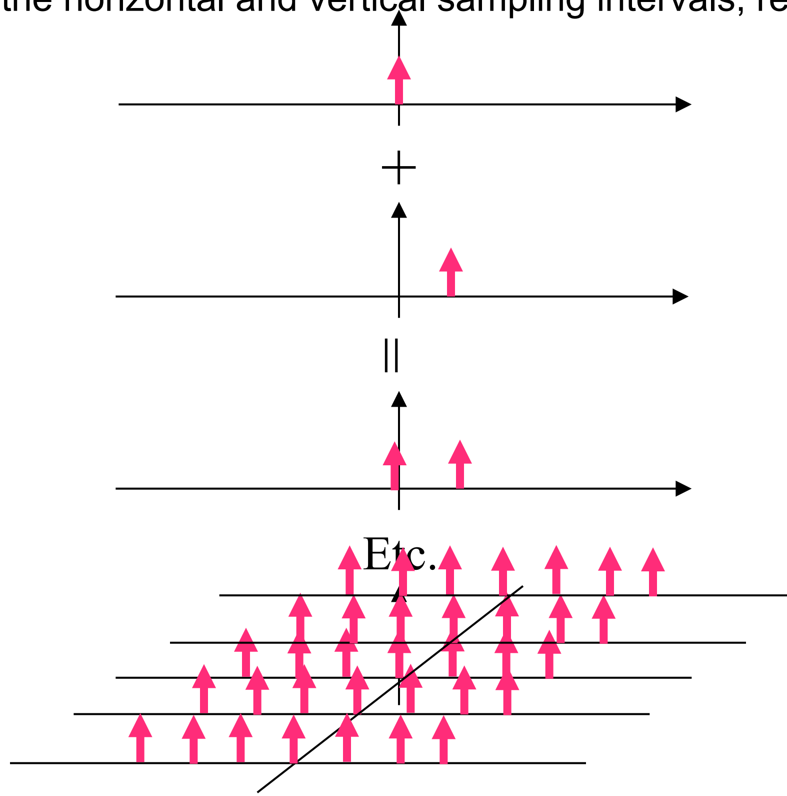
The discrete case: Discrete image sampling

The forward transform

- When the image is digitized, the irradiance is known only at a discrete set of locations.
- We think of the results as defined by a discrete grid of impulses

$$f(x,y) = wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl} \delta(x - kw, y - lh)$$

where w and h are the horizontal and vertical sampling intervals, respectively.



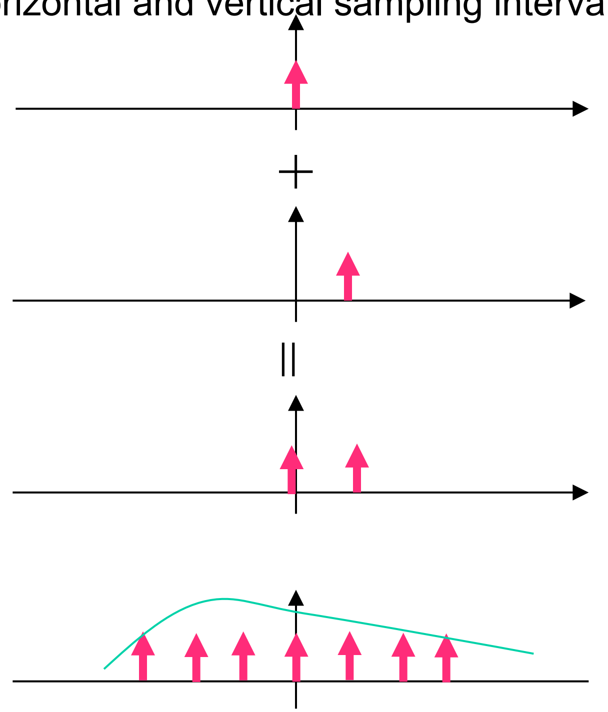
The discrete case: Discrete image sampling

The forward transform

- When the image is digitized, the irradiance is known only at a discrete set of locations.
- We think of the results as defined by a discrete grid of impulses

$$f(x,y) = wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl}\,\delta(x-kw, y-lh)$$

where w and h are the horizontal and vertical sampling intervals, respectively.



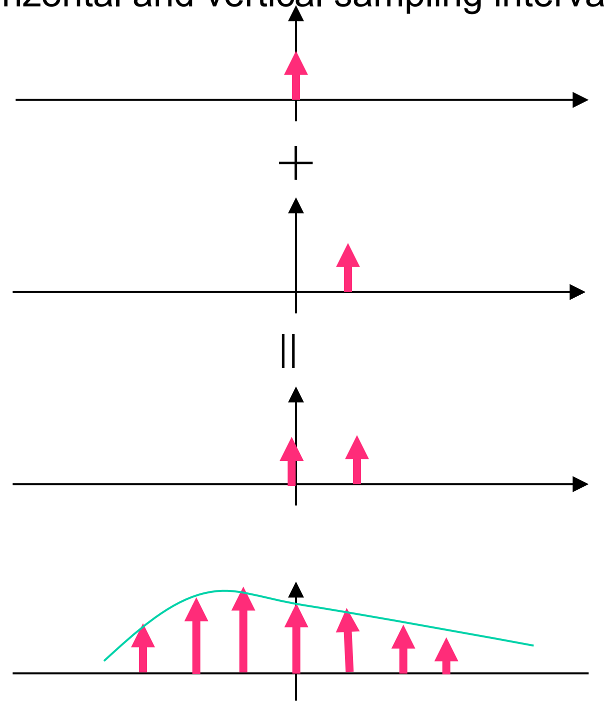
The discrete case: Discrete image sampling

The forward transform

- When the image is digitized, the irradiance is known only at a discrete set of locations.
- We think of the results as defined by a discrete grid of impulses

$$f(x,y) = wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl}\,\delta(x-kw, y-lh)$$

where w and h are the horizontal and vertical sampling intervals, respectively.



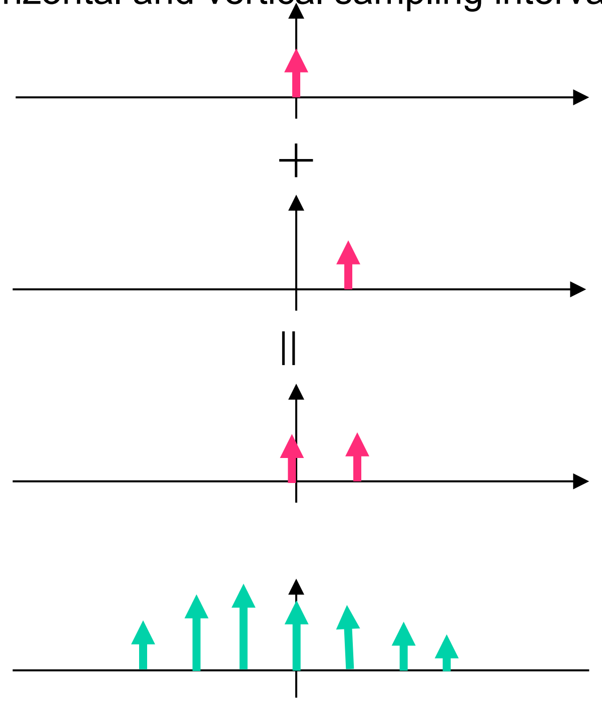
The discrete case: Discrete image sampling

The forward transform

- When the image is digitized, the irradiance is known only at a discrete set of locations.
- We think of the results as defined by a discrete grid of impulses

$$f(x,y) = wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl}\,\delta(x-kw, y-lh)$$

where w and h are the horizontal and vertical sampling intervals, respectively.



The discrete case: Discrete image sampling

The forward transform

- When the image is digitized, the irradiance is known only at a discrete set of locations.
- We think of the results as defined by a discrete grid of impulses

$$f(x,y) = wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl}\,\delta(x-kw,\, y-lh)$$

where w and h are the horizontal and vertical sampling intervals, respectively.

The discrete case: Discrete image sampling

The forward transform

- When the image is digitized, the irradiance is known only at a discrete set of locations.
- We think of the results as defined by a discrete grid of impulses

$$f(x,y) = wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl}\,\delta(x-kw, y-lh)$$

where w and h are the horizontal and vertical sampling intervals, respectively.

- The Fourier transform now becomes

$$F(u,v) \quad = \quad wh\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl}\,\delta(x-kw, y-lh)\exp[-i(ux+vy)]dxdy$$

The discrete case: Discrete image sampling

The forward transform

- When the image is digitized, the irradiance is known only at a discrete set of locations.
- We think of the results as defined by a discrete grid of impulses

$$f(x,y) = wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl} \delta(x - kw, y - lh)$$

  where w and h are the horizontal and vertical sampling intervals, respectively.

- The Fourier transform now becomes

$$\begin{aligned} F(u,v) &= wh \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl} \delta(x - kw, y - lh) \exp[-i(ux + vy)] dx dy \\ &= wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl} \exp[-i(ukw + vlh)] \end{aligned}$$

The discrete case: Discrete image sampling

The forward transform

- When the image is digitized, the irradiance is known only at a discrete set of locations.
- We think of the results as defined by a discrete grid of impulses

$$f(x,y) = wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl}\delta(x-kw, y-lh)$$

where w and h are the horizontal and vertical sampling intervals, respectively.

- The Fourier transform now becomes

$$\begin{aligned} F(u,v) &= wh\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl}\delta(x-kw, y-lh)\exp[-i(ux+vy)]dxdy \\ &= wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl} \exp[-i(ukw+vlh)] \end{aligned}$$

- This is a periodic function
  - The period in u is $2\pi / w$
  - The period in v is $2\pi / h$

The discrete case: Discrete image sampling

The forward transform

- When the image is digitized, the irradiance is known only at a discrete set of locations.
- We think of the results as defined by a discrete grid of impulses

$$f(x,y) = wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl}\delta(x-kw, y-lh)$$

  where w and h are the horizontal and vertical sampling intervals, respectively.
- The Fourier transform now becomes

$$\begin{aligned} F(u,v) &= wh\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl}\delta(x-kw, y-lh)\exp[-i(ux+vy)]dxdy \\ &= wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl} \exp[-i(ukw+vlh)] \end{aligned}$$

- This is a periodic function
  - The period in u is $2\pi / w$: consider $\cos(ukw), u = 2\pi / w \Rightarrow \cos[(2\pi / w)kw]$
  - The period in v is $2\pi / h$

The discrete case: Discrete image sampling

The forward transform

- When the image is digitized, the irradiance is known only at a discrete set of locations.
- We think of the results as defined by a discrete grid of impulses

$$f(x,y) = wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl}\delta(x-kw, y-lh)$$

where w and h are the horizontal and vertical sampling intervals, respectively.

- The Fourier transform now becomes

$$\begin{aligned} F(u,v) &= wh\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl}\delta(x-kw, y-lh)\exp[-i(ux+vy)]dxdy \\ &= wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl} \exp[-i(ukw+vlh)] \end{aligned}$$

- This is a periodic function
  - The period in u is $2\pi / w$: consider $\cos(ukw), u = 2\pi / w \Rightarrow \cos[(2\pi / \cancel{w})k\cancel{w}] = \cos(2\pi k)$
  - The period in v is $2\pi / h$

The discrete case: Discrete image sampling

The forward transform

- When the image is digitized, the irradiance is known only at a discrete set of locations.
- We think of the results as defined by a discrete grid of impulses

$$f(x,y) = wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl}\delta(x-kw, y-lh)$$

where w and h are the horizontal and vertical sampling intervals, respectively.

- The Fourier transform now becomes

$$\begin{aligned} F(u,v) &= wh\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl}\delta(x-kw, y-lh)\exp[-i(ux+vy)]dxdy \\ &= wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl} \exp[-i(ukw+vlh)] \end{aligned}$$

- This is a periodic function
  - The period in u is $2\pi / w$: consider $\cos(ukw), u = 2\pi / w \Rightarrow \cos[(2\pi / w)kw] = \cos(2\pi k)$
  - The period in v is $2\pi / h$: and similarly for v

The discrete case: Discrete image sampling

The forward transform

- When the image is digitized, the irradiance is known only at a discrete set of locations.
- We think of the results as defined by a discrete grid of impulses

$$f(x,y) = wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl}\delta(x-kw, y-lh)$$

  where w and h are the horizontal and vertical sampling intervals, respectively.
- The Fourier transform now becomes

$$\begin{aligned} F(u,v) &= wh\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl}\delta(x-kw, y-lh)\exp[-i(ux+vy)]dxdy \\ &= wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl} \exp[-i(ukw+vlh)] \end{aligned}$$

- This is a periodic function
  - The period in u is $2\pi / w$
  - The period in v is $2\pi / h$
  - We can ignore that part of $F(u,v)$ for $|u| > \pi / w$ and $|v| > \pi / h$

The discrete case: Discrete image sampling

The forward transform

- When the image is digitized, the irradiance is known only at a discrete set of locations.
- We think of the results as defined by a discrete grid of impulses

$$f(x,y) = wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl}\delta(x-kw, y-lh)$$

  where w and h are the horizontal and vertical sampling intervals, respectively.

- The Fourier transform now becomes

$$\begin{aligned} F(u,v) &= wh\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl}\delta(x-kw, y-lh)\exp[-i(ux+vy)]dxdy \\ &= wh \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{kl}\exp[-i(ukw+vlh)] \end{aligned}$$

- This is a periodic function
  - The period in u is $2\pi / w$
  - The period in v is $2\pi / h$
  - We can ignore that part of $F(u,v)$ for $|u| > \pi / w$ and $|v| > \pi / h$
- **Moral of the story:** By sampling the function in the spatial domain, we have circumscribed the information in the frequency domain.

The discrete case: Discrete image sampling

The inverse transform

- Let

$$\widetilde{F}(u,v) = \begin{cases} F(u,v), & |u| \le \pi/w \text{ and } |v| \le \pi/h \\ 0, & |u| > \pi/w \text{ or } |v| > \pi/h \end{cases}$$

The discrete case: Discrete image sampling

The inverse transform

- Let

$$\widetilde{F}(u,v) = \begin{cases} F(u,v), & |u| \le \pi/w \text{ and } |v| \le \pi/h \\ 0, & |u| > \pi/w \text{ or } |v| > \pi/h \end{cases}$$

- The corresponding inverse transform is

$$\widetilde{f}(x,y) = \frac{1}{4\pi^2} \int_{-\pi/h}^{\pi/h} \int_{-\pi/w}^{\pi/w} \widetilde{F}(u,v) \exp[i(ux + vy)] du dv$$

The discrete case: Discrete image sampling

The inverse transform

- Let

$$\widetilde{F}(u,v) = \begin{cases} F(u,v), & |u| \le \pi/w \text{ and } |v| \le \pi/h \\ 0, & |u| > \pi/w \text{ or } |v| > \pi/h \end{cases}$$

- The corresponding inverse transform is

$$\widetilde{f}(x,y) = \frac{1}{4\pi^2}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w} \widetilde{F}(u,v)\exp[i(ux+vy)]dudv$$

- We can write this as

$$\widetilde{f}(x,y) = \frac{wh}{4\pi^2}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w} \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\exp[-i(ukw+vlh)]\exp[i(ux+vy)]dudv$$

The discrete case: Discrete image sampling

The inverse transform

- We continue with the evaluation of our integral

$$\tilde{f}(x,y) = \frac{wh}{4\pi^2}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\exp[-i(ukw+vlh)]\exp[i(ux+vy)]dudv$$

The discrete case: Discrete image sampling

The inverse transform

- We continue with the evaluation of our integral

$$\tilde{f}(x,y) = \frac{wh}{4\pi^2}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\exp[-i(ukw+vlh)]\exp[i(ux+vy)]\,du\,dv$$

pull f out of integral, group by u & v

$$= \frac{wh}{4\pi^2}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w} \exp\{i[u(x-kw)+v(y-lh)]\}\,du\,dv$$

The discrete case: Discrete image sampling

The inverse transform

- We continue with the evaluation of our integral

$$\tilde{f}(x,y) = \frac{wh}{4\pi^2}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\exp[-i(ukw+vlh)]\exp[i(ux+vy)]dudv$$

$$= \frac{wh}{4\pi^2}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w}\exp\{i[u(x-kw)+v(y-lh)]\}dudv$$

write exponential as product of u & v terms

$$= \frac{wh}{4\pi^2}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w}\exp i[u(x-kw)]\exp i[v(y-lh)]dudv$$

The discrete case: Discrete image sampling

The inverse transform

- We continue with the evaluation of our integral

$$\tilde{f}(x,y) = \frac{wh}{4\pi^2}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\exp[-i(ukw+vlh)]\exp[i(ux+vy)]dudv$$

$$= \frac{wh}{4\pi^2}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w}\exp\{i[u(x-kw)+v(y-lh)]\}dudv$$

$$= \frac{wh}{4\pi^2}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w}\exp i[u(x-kw)]\exp i[v(y-lh)]dudv$$

expand exponential; cancel sinusoidal terms

$$= \frac{wh}{4\pi^2}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w}\cos u(x-kw)\cos v(y-lh)dudv$$

The discrete case: Discrete image sampling

The inverse transform

- We continue with the evaluation of our integral

$$\tilde{f}(x,y) = \frac{wh}{4\pi^2}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\exp[-i(ukw+vlh)]\exp[i(ux+vy)]dudv$$

$$= \frac{wh}{4\pi^2}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w} \exp\{i[u(x-kw)+v(y-lh)]\}dudv$$

$$= \frac{wh}{4\pi^2}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w} \exp i[u(x-kw)]\exp i[v(y-lh)]dudv$$

$$= \frac{wh}{4\pi^2}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w}\cos u(x-kw)\cos v(y-lh)dudv$$

evaluate integral; perform algebra

$$= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\frac{\sin\pi(x/w-k)}{\pi(x/w-k)}\frac{\sin\pi(y/h-l)}{\pi(y/h-l)}$$

The discrete case: Discrete image sampling

Evaluate the integral and perform algebra

- Remark

$$\frac{w}{2\pi}\int_{-\pi/w}^{\pi/w}\cos u(x-kw)\,du$$

$$=\frac{\cancel{w}}{2\pi}\frac{[\sin u(x-kw)]_{-\pi/w}^{\pi/w}}{\cancel{w}(x/w-k)}$$

$$=\frac{1}{2\pi}\frac{\sin[(\pi/w)(x-kw)]-\sin[(-\pi/w)(x-kw)]}{(x/w-k)}$$

$$=\frac{1}{\cancel{2}\pi}\frac{\cancel{2}\sin[(\pi/w)(x-kw)]}{(x/w-k)}$$

$$\frac{\sin\pi(x/w-k)}{\pi(x/w-k)}$$

The discrete case: Discrete image sampling

The inverse transform

- We continue with the evaluation of our integral

$$\tilde{f}(x,y) = \frac{wh}{4\pi^2}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\exp[-i(ukw+vlh)]\exp[i(ux+vy)]dudv$$

$$= \frac{wh}{4\pi^2}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w}\exp\{i[u(x-kw)+v(y-lh)]\}dudv$$

$$= \frac{wh}{4\pi^2}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w}\exp i[u(x-kw)]\exp i[v(y-lh)]dudv$$

$$= \frac{wh}{4\pi^2}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w}\cos u(x-kw)\cos v(y-lh)dudv$$

evaluate integral; perform algebra

$$= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\frac{\sin\pi(x/w-k)}{\pi(x/w-k)}\frac{\sin\pi(y/h-l)}{\pi(y/h-l)}$$

The discrete case: Discrete image sampling

The inverse transform

- We continue with the evaluation of our integral

$$
\begin{aligned}
\tilde{f}(x,y) &= \frac{wh}{4\pi^2}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\exp[-i(ukw+vlh)]\exp[i(ux+vy)]dudv \\
&= \frac{wh}{4\pi^2}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w}\exp\{i[u(x-kw)+v(y-lh)]\}dudv \\
&= \frac{wh}{4\pi^2}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w}\exp i[u(x-kw)]\exp i[v(y-lh)]dudv \\
&= \frac{wh}{4\pi^2}\sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\int_{-\pi/h}^{\pi/h}\int_{-\pi/w}^{\pi/w}\cos u(x-kw)\cos v(y-lh)dudv \\
&= \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} f_{kl}\frac{\sin\pi(x/w-k)}{\pi(x/w-k)}\frac{\sin\pi(y/h-l)}{\pi(y/h-l)}
\end{aligned}
$$

- We notice that the final expression is simply an interpolation of the sampled image points f_{kl}
- Apparently, the fact that $F(u,v)$ was zero beyond a certain range of u and v has made it possible to reconstruct the image from a discrete set of samples.

The discrete case: The sampling theorem

What has been shown

- We have just seen that a function that is bandlimited (i.e., has frequency components evaluating to zero beyond some range) is fully specified by samples on a regular grid.
- This result is known as the sampling theorem.
- If $F(u,v)=0$ for $|u| > \pi / w$ or $|v| > \pi / h$, then $f(x,y)$ can be recovered from the set $f(kw,lh)$ for all integers k and l.

The discrete case: The sampling theorem

What has been shown

- We have just seen that a function that is bandlimited (i.e., has frequency components evaluating to zero beyond some range) is fully specified by samples on a regular grid.
- This result is known as the sampling theorem.
- If $F(u,v)=0$ for $|u|>\pi/w$ or $|v|>\pi/h$, then $f(x,y)$ can be recovered from the set $f(kw,lh)$ for all integers k and l.

The sampling interval

- If only frequencies less that B occur, then the sampling interval can be a large as $d=\pi/B$

The discrete case: The sampling theorem

What has been shown

- We have just seen that a function that is bandlimited (i.e., has frequency components evaluating to zero beyond some range) is fully specified by samples on a regular grid.
- This result is known as the sampling theorem.
- If $F(u,v)=0$ for $|u| > \pi / w$ or $|v| > \pi / h$, then $f(x,y)$ can be recovered from the set $f(kw,lh)$ for all integers k and l.

The sampling interval

- If only frequencies less that B occur, then the sampling interval can be a large as $d = \pi / B$
- Stated differently, the sampling interval should be less than $\lambda / 2$, with λ the wavelength of highest frequency present.

$$u\,(x+\lambda) = ux + 2\pi \;\rightarrow u = \frac{2\pi}{\lambda}$$

$$d = \frac{\pi}{B} \rightarrow u = \pi/d$$

$$\frac{\pi}{\mathrm{d}} = \frac{2\pi}{\lambda} \rightarrow d = \frac{\lambda}{2}$$

The discrete case: The sampling theorem

What has been shown

- We have just seen that a function that is bandlimited (i.e., has frequency components evaluating to zero beyond some range) is fully specified by samples on a regular grid.
- This result is known as the sampling theorem.
- If $F(u,v)=0$ for $|u|>\pi/w$ or $|v|>\pi/h$, then $f(x,y)$ can be recovered from the set $f(kw,lh)$ for all integers k and l.

The sampling interval

- If only frequencies less that B occur, then the sampling interval can be a large as $d=\pi/B$
- Stated differently, the sampling interval should be less than $\lambda/2$, with λ the wavelength of the highest frequency present.
- If d is the sampling interval, then our result can be expressed in terms of the Nyquist frequency, π/d
- The signal should contain frequencies only up to the Nyquist frequency.

The discrete case: The sampling theorem

What has been shown

- We have just seen that a function that is bandlimited (i.e., has frequency components evaluating to zero beyond some range) is fully specified by samples on a regular grid.
- This result is known as the sampling theorem.
- If $F(u,v)=0$ for $|u| > \pi / w$ or $|v| > \pi / h$, then $f(x,y)$ can be recovered from the set $f(kw,lh)$ for all integers k and l.

The sampling interval

- If only frequencies less that B occur, then the sampling interval can be a large as $d = \pi / B$
- Stated differently, the sampling interval should be less than $\lambda / 2$, with λ the wavelength of the highest frequency present.
- If d is the sampling interval, then our result can be expressed in terms of the Nyquist frequency, π / d
- The signal should contain frequencies only up to the Nyquist frequency.

Aliasing

- Frequencies higher than the Nyquist frequency, when sampled, look no different than frequencies within the acceptable limit.
- They will be aliased as lower frequencies in the result.

The discrete case: The sampling theorem

What has been shown

- We have just seen that a function that is bandlimited (i.e., has frequency components evaluating to zero beyond some range) is fully specified by samples on a regular grid.
- This result is known as the sampling theorem.
- If $F(u,v)=0$ for $|u|>\pi/w$ or $|v|>\pi/h$, then $f(x,y)$ can be recovered from the set $f(kw,lh)$ for all integers k and l.

The sampling interval

- If only frequencies less that B occur, then the sampling interval can be a large as $d=\pi/B$
- Stated differently, the sampling interval should be less than $\lambda/2$, with λ the wavelength of the highest frequency present.
- If d is the sampling interval, then our result can be expressed in terms of the Nyquist frequency, π/d
- The signal should contain frequencies only up to the Nyquist frequency.

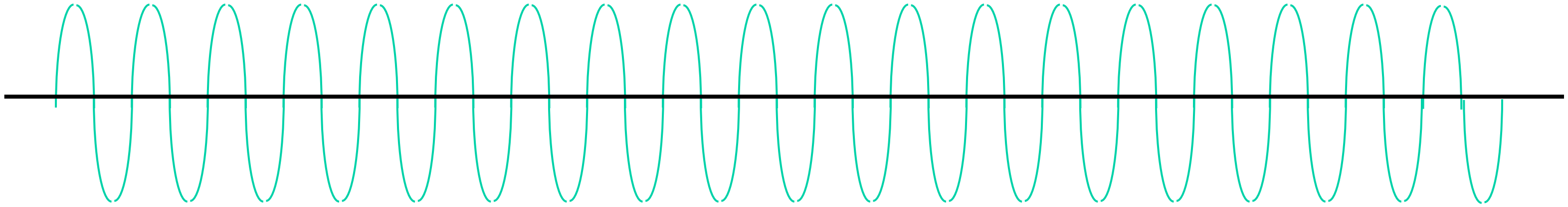
Aliasing

- Frequencies higher than the Nyquist frequency, when sampled, look no different than frequencies within the acceptable limit.
- They will be aliased as lower frequencies in the result.

Remark

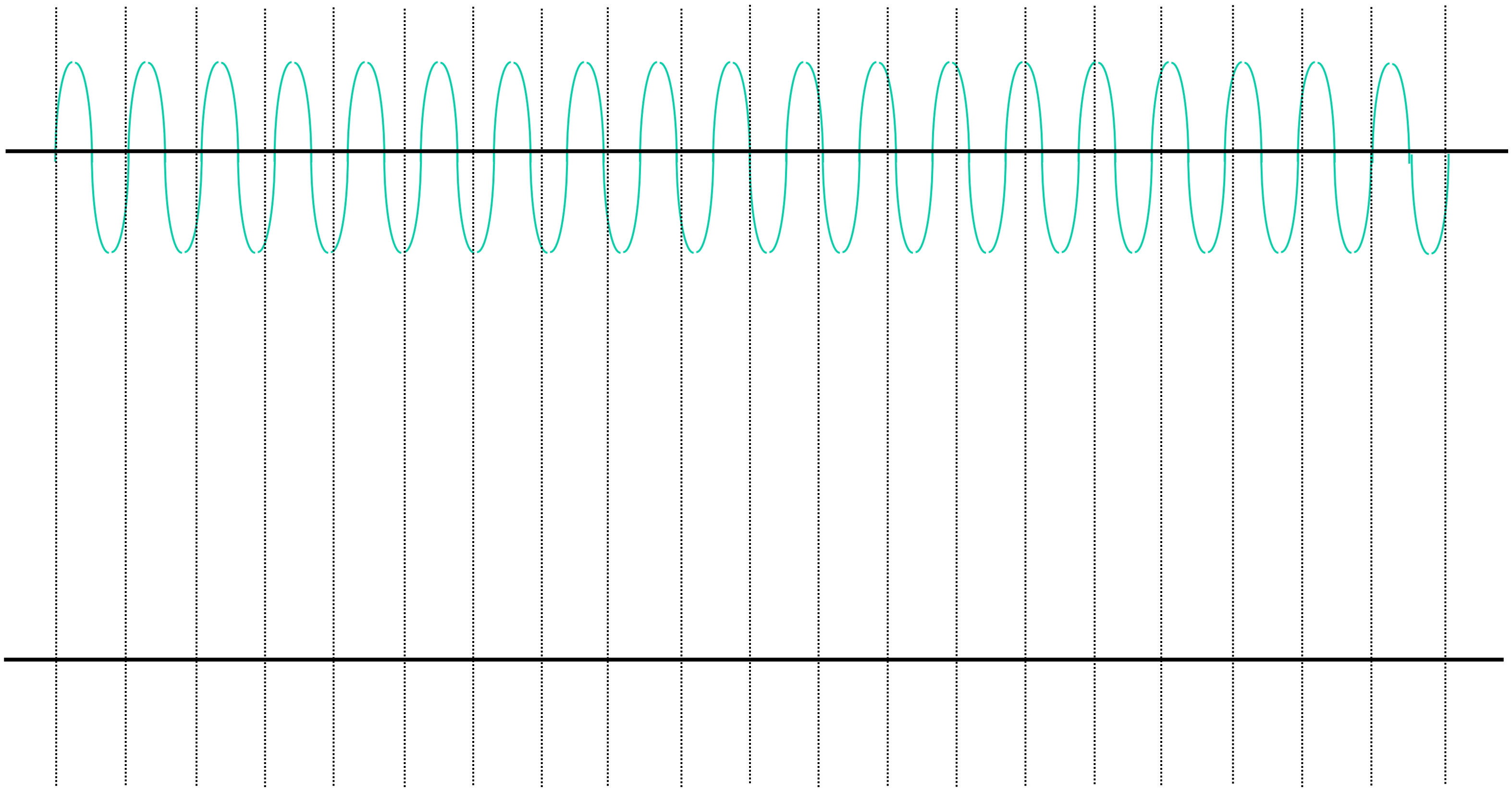
- In order to avoid certain singular situations that can occur when sampling at the minimally defined interval we typically sample at a smaller interval.
- Indeed, in the presence of significant noise, it is prudent to use a sampling interval a factor of 5 or 10 smaller than the minimally defined.

The discrete case: Aliasing example



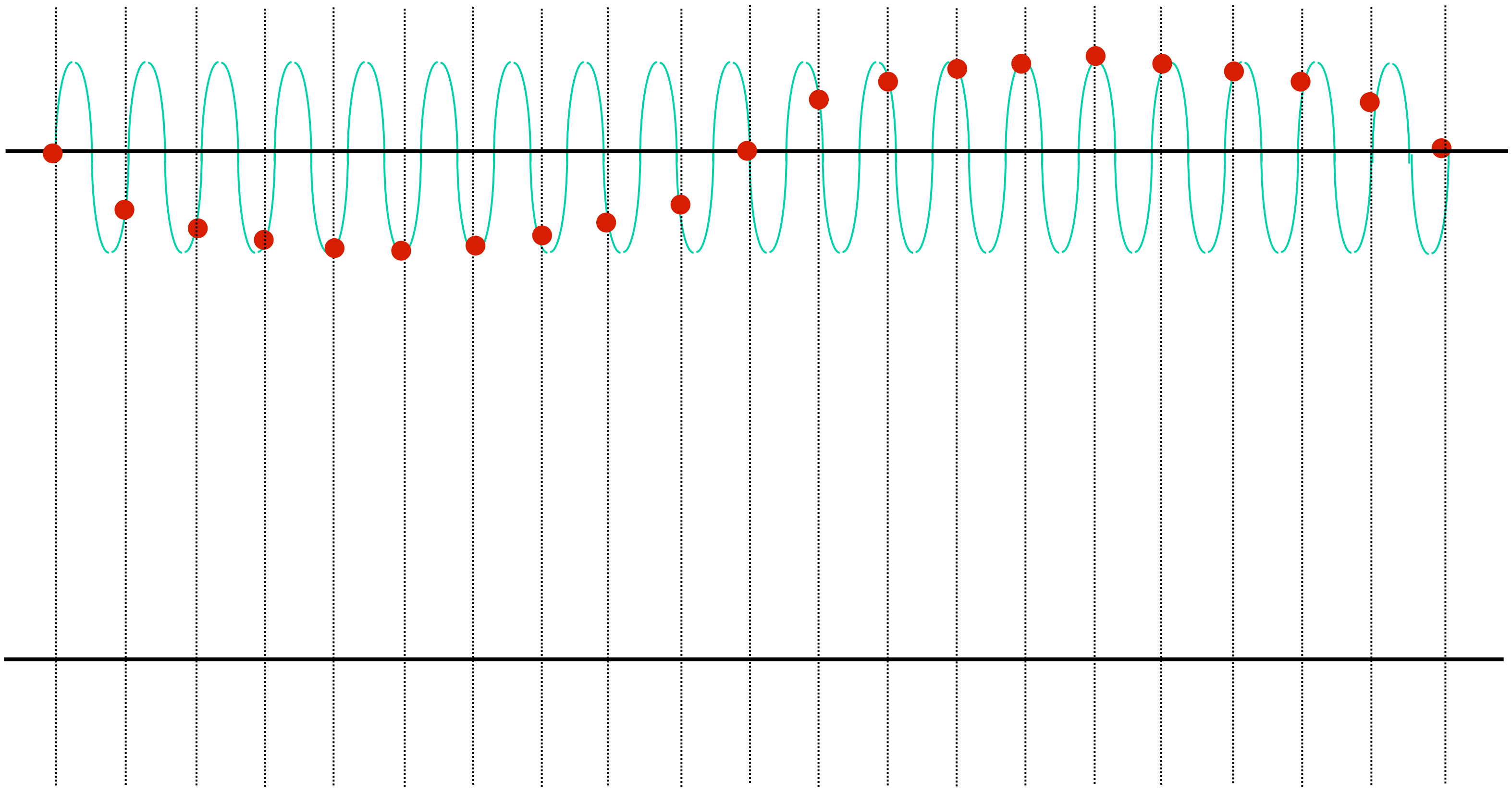
Consider: A sinusoidal pattern.

The discrete case: Aliasing example



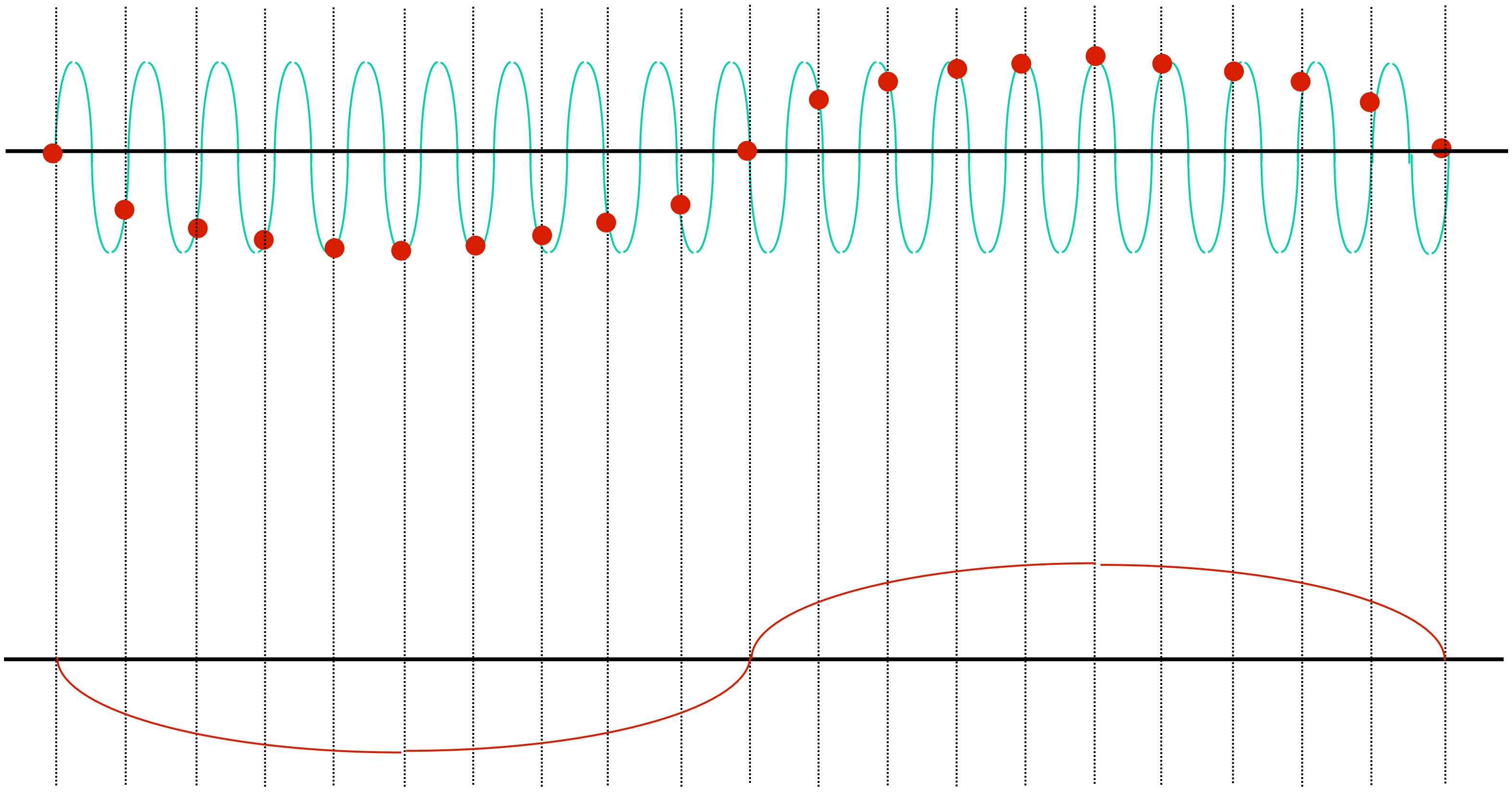
Suppose: That the sinusoid is spatially sampled at an interval greater than that of half the wavelength.

The discrete case: Aliasing example



Notice: That along the sampled points another sinusoid becomes apparent

The discrete case: Aliasing example



Conclusion: For the dotted sampling grid, the higher frequency sinusoid aliases to the lower frequency sinusoid

The discrete case: The discrete Fourier transform

Definitions

- Let an image be specified by the values f_{kl} of $f(x,y)$ at points (kw,lh) for $k=0,1,\ldots,M$-1 and $l=0,1,\ldots,N$-1.
- The discrete Fourier transform is then given as

$$F_{mn} = \sum_{k=0}^{M-1}\sum_{l=0}^{N-1} f_{kl} \exp\left[-2\pi i\left(\frac{km}{M}+\frac{ln}{N}\right)\right]$$

- The inverse transform is given as

$$f_{kl} = \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} F_{mn} \exp\left[2\pi i\left(\frac{km}{M}+\frac{ln}{N}\right)\right]$$

The discrete case: The discrete convolution

Definition

- Let the functions $f(x,y)$ and $h(x,y)$ be specified by their values at a discrete grid of points as f_{ij} and h_{ij} , respectively.
- We define the discrete convolution g_{ij} of f and h as

$$g_{ij} = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{i-k,j-l} h_{k,l}$$

The discrete case: The discrete convolution

Definition

- Let the functions $f(x,y)$ and $h(x,y)$ be specified by their values at a discrete grid of points as f_{ij} and h_{ij} , respectively.
- We define the discrete convolution g_{ij} of f and h as

$$g_{ij} = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{i-k,j-l} h_{k,l}$$

- Notice that this is consistent with our earlier continuous definition of convolution as

$$g(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-\xi, y-\eta) h(\xi,\eta) d\xi d\eta$$

with i, j taking the roles of x,y and k,l taking the roles of ξ, η

The discrete case: Neighborhoods

Implementation of discrete operations

- To apply many of the operations that we have described (convolution, etc.), we must sample underlying continuous functions.
- Just as we have discussed the sampling of images,
- we also will need to sample point spread functions, templates for correlation, etc.
- The idea is really the same, we sample the continuous representation to get a discrete counterpart.

The discrete case: Neighborhoods

Implementation of discrete operations

- To apply many of the operations that we have described (convolution, etc.), we must sample underlying continuous functions.
- Just as we have discussed the sampling of images,
- we also will need to sample point spread functions, templates for correlation, etc.
- The idea is really the same, we sample the continuous representation to get a discrete counterpart.
- We will sometimes refer to these discrete counterparts as (digital) masks or stencils.
- We will refer to the individual (numerical) elements within the masks as taps.
- For example, one reasonable sampling of the Gaussian point spread function (with unit standard deviation) yields the following mask

$$\frac{1}{16^2}\begin{bmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 24 & 16 & 4 \\ 6 & 24 & 36 & 24 & 6 \\ 4 & 16 & 24 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{bmatrix}$$

The discrete case: Pseudocode

Procedure

- Input: f an N x M image and h an m x m convolution mask, $m < N, M$.
- Output: g an N x M image that is the convolution of f with h.

The discrete case: Pseudocode

Procedure

- Input: f an N x M image and h an m x m convolution mask, $m < N, M$.
- Output: g an N x M image that is the convolution of f with h.
- For all i, j

$$g_{ij} = \sum_{k=-m/2}^{m/2} \sum_{l=-m/2}^{m/2} f_{i-k,j-l} h_{k,l}$$

with $m/2$ integer division (i.e., 3/2 = 1).

The discrete case: Pseudocode

Procedure

- Input: f an N x M image and h an m x m convolution mask, $m < N, M$.
- Output: g an N x M image that is the convolution of f with h.
- For all i, j

$$g_{ij} = \sum_{k=-m/2}^{m/2} \sum_{l=-m/2}^{m/2} f_{i-k,j-l} h_{k,l}$$

with $m/2$ integer division (i.e., 3/2 = 1).

Remark

- At the image borders, where the mask does not fit within the image, there are several choices
  1. Design special masks that are adapted to such configurations.
  2. Assume that values outside the image are some constant value, e.g., 0.
  3. Reflect the image about its borders.

The discrete case: A refinement

Separability

- Notice that

$$\frac{1}{16^2}\begin{bmatrix}1 & 4 & 6 & 4 & 1\\ 4 & 16 & 24 & 16 & 4\\ 6 & 24 & 36 & 24 & 6\\ 4 & 16 & 24 & 16 & 4\\ 1 & 4 & 6 & 4 & 1\end{bmatrix} = (1/16)\begin{bmatrix}1\\ 4\\ 6\\ 4\\ 1\end{bmatrix}(1/16)\begin{bmatrix}1 & 4 & 6 & 4 & 1\end{bmatrix}$$

- Correspondingly, the 2D convolution can be separated into 2 1D convolutions.
- A separable implementation yields increased computational efficiency, e.g., for convolution with an N x N mask in 2D requires N^2 operations at each point, while 2 1D convolutions require only 2 N operations.
- Remark: Here, separability arises from the fact that the underlying Gaussian PSF has the form $\exp(a+b) = \exp(a)\exp(b)$.

The discrete case: Another refinement

Steerability

- Recall that the directional derivative along some direction $\mathbf{v} = (\cos a, \sin a)$ can be had as

$$\mathbf{v} \bullet \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right) f(x, y)$$

or in terms of convolution as

$$= \mathbf{v} \bullet \left(\delta_x * f(x, y), \delta_y * f(x, y) \right)$$

- In words: The directional derivative along any direction can be computed as the weighted sum of the partial derivatives along just two directions.
- We say that the two convolutions are steered according to the coefficients $(\cos a, \sin a)$ specified by $\mathbf{v}$ to yield the final result.
- Computational advantage can be had as we can precompute the needed interim basis results and subsequently combine them according to our subsequent needs.
- Remark: Steerability can be invoked when ever the desired function can be represented as set of basis functions.

The discrete case: Example revisited

Noise corrupted image

Gaussian blurred image

Remark: The depicted transformation was accomplished via the just described convolution algorithm using the Gaussian PSF mask given a few slides back.

Outline

Basics

The Fourier transform

Local operators

Restoration and enhancement

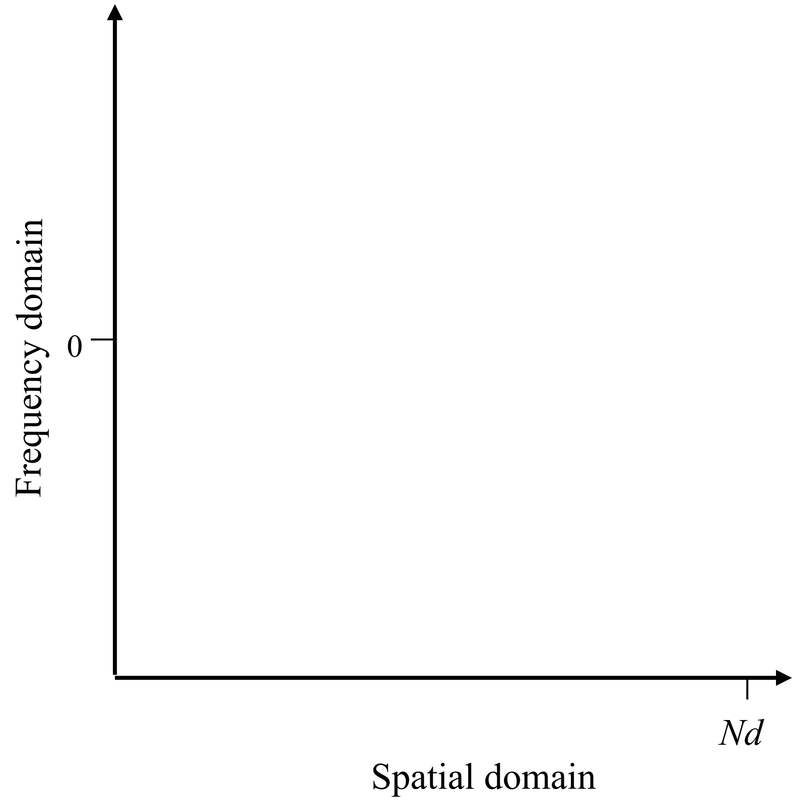
The discrete case

Local scale and orientation

Local scale: Motivation

Where we stand

- We have discussed two different ways to represent an image
  - The spatial domain
  - The frequency domain



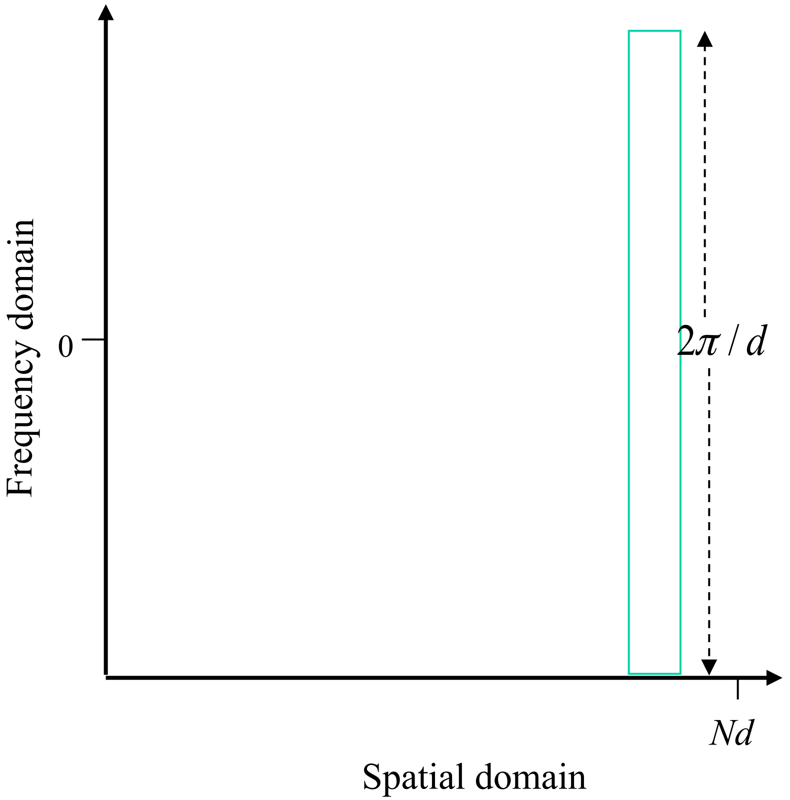
Local scale: Motivation

Where we stand

- We have discussed two different ways to represent an image
  - The spatial domain
  - The frequency domain

Spatial domain

- We know spatial position with a precision of the sampling interval, d.
- Frequency resolution lost: We only can identify the frequency to be within the range $\pm\pi/d$.
- We know the local image intensity (e.g., irradiance), but have little knowledge of frequency structure.



Local scale: Motivation

Where we stand

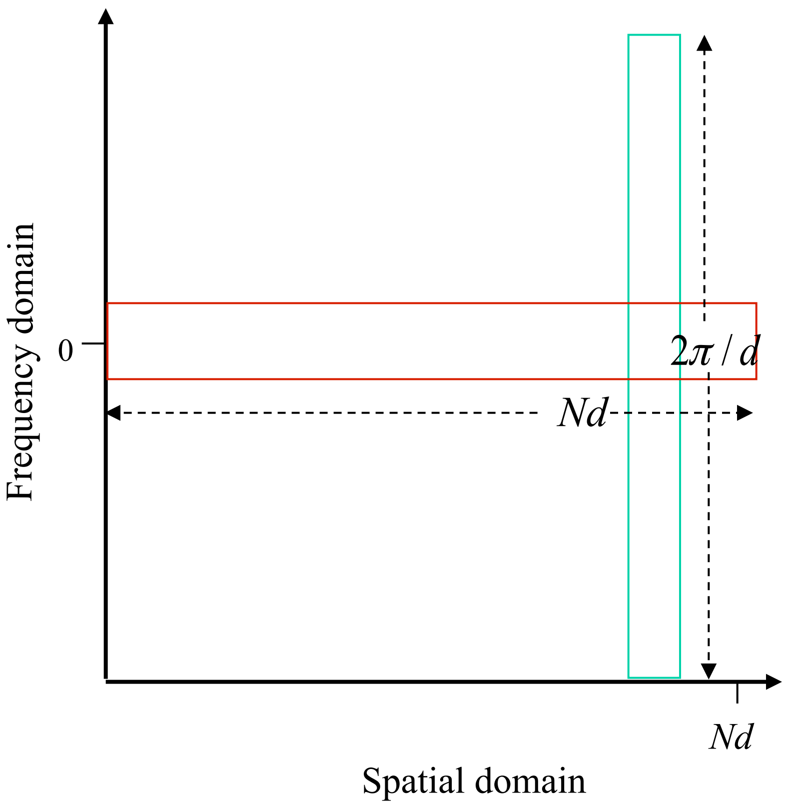
- We have discussed two different ways to represent an image
  - The spatial domain
  - The frequency domain

Spatial domain

- We know spatial position with a precision of the sampling interval, d.
- Frequency resolution lost: We only can identify the frequency to be within the range $\pm\pi / d$.
- We know the local image intensity (e.g., irradiance), but have little knowledge of frequency structure.

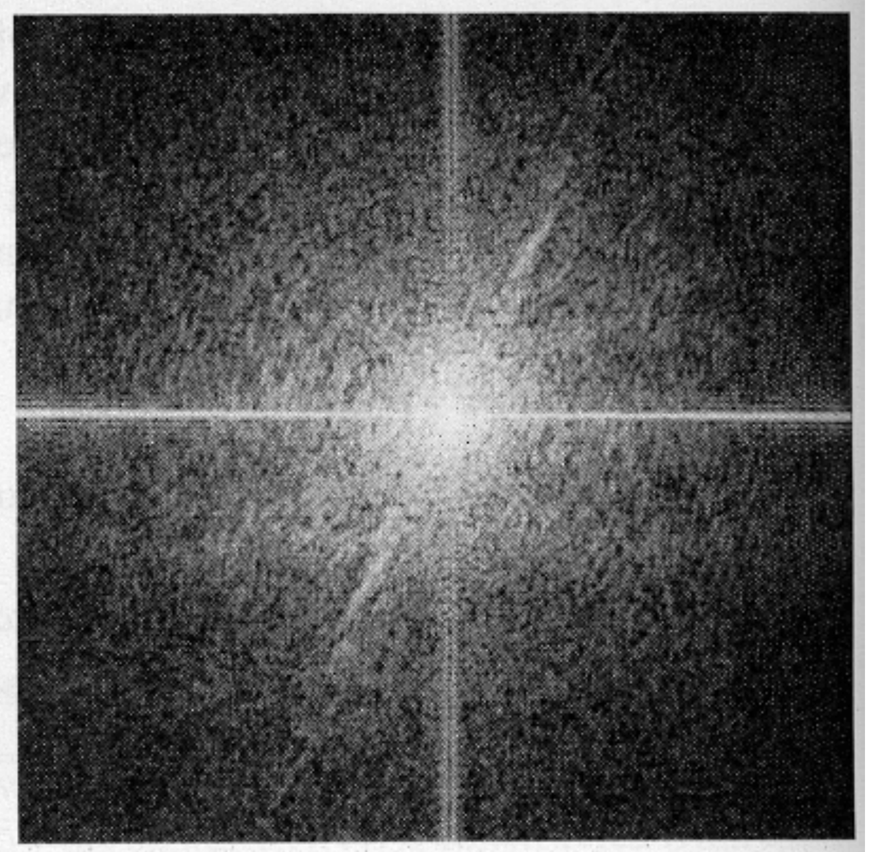
Frequency domain

- We can resolve the frequency content with precision.
- Spatial position is lost: We only can identify the position to be within the range Nd.
- We know the frequency structure, but cannot localize it spatially.



Local scale: Motivation

Source image (J. Fourier)

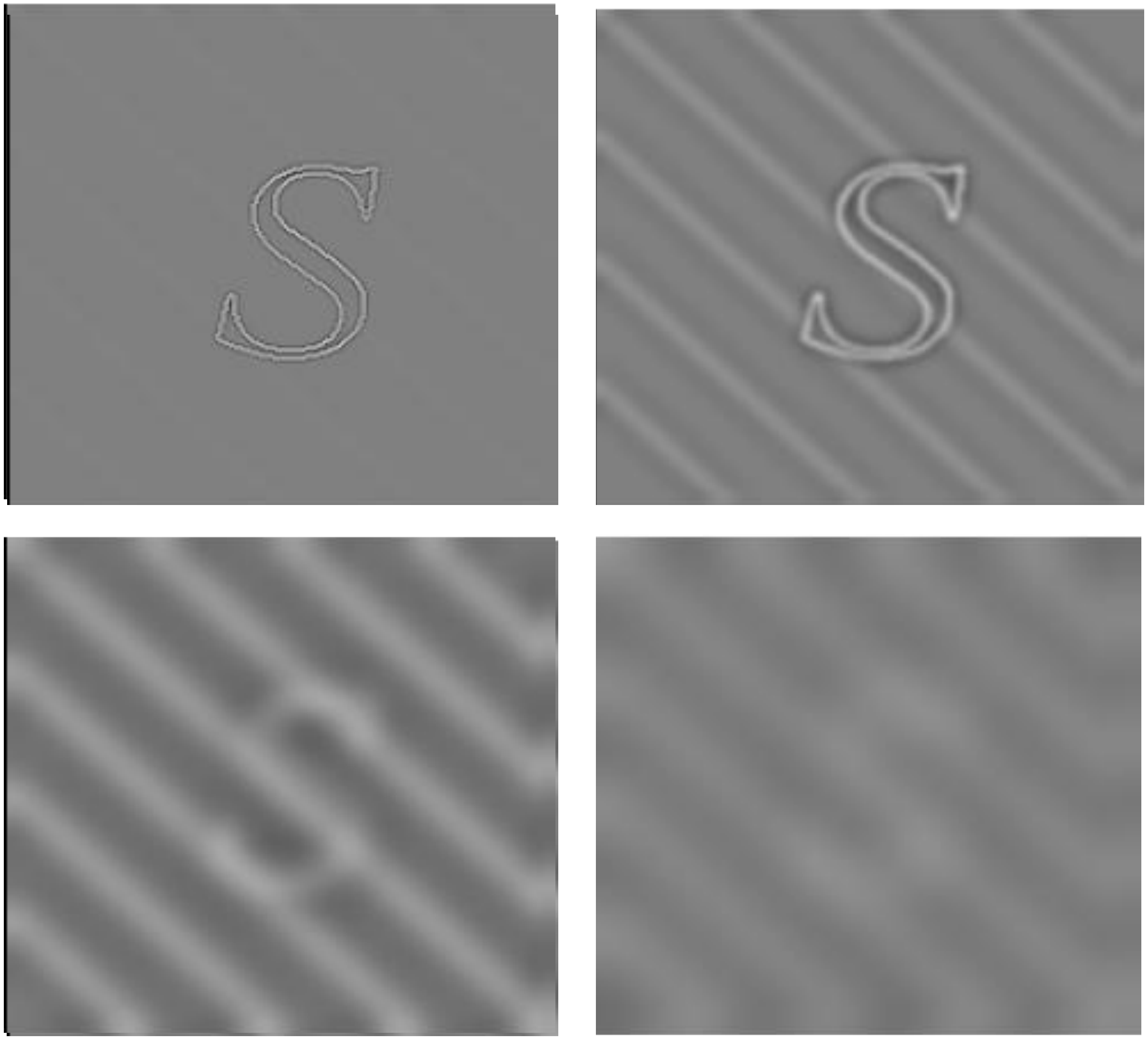


Power spectrum

Local scale: Motivation

Concept: Provide a local representation of frequency content.

Advantage: Provides a principled parsing of the local image structure.



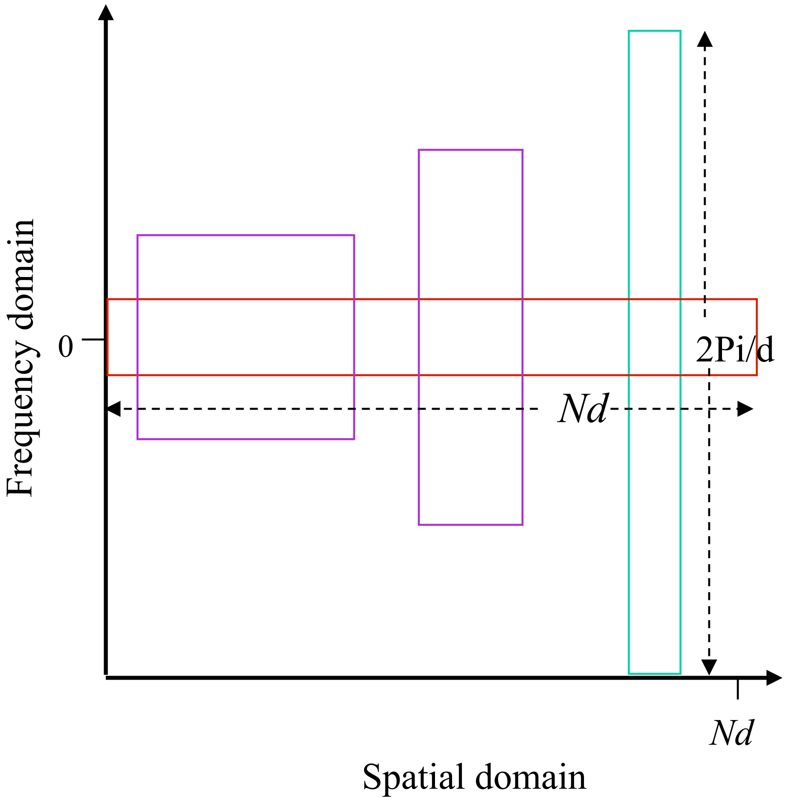
Local scale: Motivation

Where we stand

- We have discussed two different ways to represent an image
  - The spatial domain
  - The frequency domain

We seek a compromise

- We desire a joint representation that allows us to capture the range of scales present locally in the image.
- Refer to such representations as multiscale or multiresolution.



Local scale: Windowed Fourier transform

Intuition

- Imagine breaking an image up into a set of tiles.
- Apply the Fourier transform to each tile individually.
- Then perhaps we have captured a spatial frequency representation that is local to each tile.

Local scale: Windowed Fourier transform

Intuition

- Imagine breaking an image up into a set of tiles.
- Apply the Fourier transform to each tile individually.
- Then perhaps we have captured a spatial frequency representation that is local to each tile.

Formalization

- We consider application of the Fourier transform, $F(u,v)$, within a window, $w(x,y)$, of the image and move the window across the image.

Local scale: Windowed Fourier transform

Intuition

- Imagine breaking an image up into a set of tiles.
- Apply the Fourier transform to each tile individually.
- Then perhaps we have captured a spatial frequency representation that is local to each tile.

Formalization

- We consider application of the Fourier transform, $F(u,v)$, within a window, $w(x,y)$, of the image and move the window across the image.
- Generally useful properties for the window include
  - It has a maximum at its center.
  - It is (circularly) symmetric about the origin.
  - It decreases monotonically with distance from the center.

Local scale: Windowed Fourier transform

Intuition

- Imagine breaking an image up into a set of tiles.
- Apply the Fourier transform to each tile individually.
- Then perhaps we have captured a spatial frequency representation that is local to each tile.

Formalization

- We consider application of the Fourier transform, $F(u,v)$, within a window, $w(x,y)$, of the image and move the window across the image.
- Generally useful properties for the window include
  - It has a maximum at its center.
  - It is (circularly) symmetric about the origin.
  - It decreases monotonically with distance from the center.
- Given such a window, we define the windowed Fourier transform as

$$F_w(x,y,u_0,v_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)w(\xi - x,\eta - y)\exp[-i(u_0\xi + v_0\eta)]d\xi d\eta$$

- So, we associate a local frequency decomposition with each image spatial position.

Local scale: Windowed Fourier transform

A closer look

- We notice that our windowed Fourier transform

$$F_w(x, y, u_0, v_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta) w(\xi - x, \eta - y) \exp[-i(u_0\xi + v_0\eta)] d\xi d\eta$$

looks almost like a convolution.

Local scale: Windowed Fourier transform

A closer look

- We notice that our windowed Fourier transform

$$F_w(x, y, u_0, v_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta) w(\xi - x, \eta - y) \exp[-i(u_0\xi + v_0\eta)] d\xi d\eta$$

  looks almost like a convolution.
- We follow this observation to rewrite F_w as

$$F_w(x, y, u_0, v_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{matrix} f(\xi,\eta) w(x-\xi, y-\eta) \exp\{i[u_0(x-\xi) + v_0(y-\eta)]\} \\ \times \exp[-i(u_0 x + v_0 y)] d\xi d\eta \end{matrix}$$

  where we have made use of the fact that $w(x)=w(-x)$ by symmetry.

Local scale: Windowed Fourier transform

A closer look

- We notice that our windowed Fourier transform

$$F_w(x,y,u_0,v_0)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)w(\xi-x,\eta-y)\exp[-i(u_0\xi+v_0\eta)]d\xi d\eta$$

  looks almost like a convolution.
- We follow this observation to rewrite F_w as

$$F_w(x,y,u_0,v_0)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \begin{array}{l} f(\xi,\eta)w(x-\xi,y-\eta)\exp\{i[u_0(x-\xi)+v_0(y-\eta)]\} \\ \times\exp[-i(u_0x+v_0y)]d\xi d\eta \end{array}$$

  where we have made use of the fact that $w(x)=w(-x)$ by symmetry.
- We now have exactly the form of a convolution, in particular

$$f(x,y)*w(x,y)\exp[i(u_0x+v_0y)]$$

  with the inclusion of an additional phase component

$$\exp[-i(u_0x+v_0y)]$$

Local scale: Windowed Fourier transform

A closer look (continued)

- Following our usual plan of attack, we choose to understand the operation of the convolution via calculation of the MTF of the point spread function,

$$w(x, y)\exp[i(u_0 x + v_0 y)]$$

Local scale: Windowed Fourier transform

A closer look (continued)

- Following our usual plan of attack, we choose to understand the operation of the convolution via calculation of the MTF of the point spread function,

$$w(x,y)\exp[i(u_0 x + v_0 y)]$$

- Correspondingly, we consider

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(x,y)\exp[i(u_0 x + v_0 y)]\exp[-i(ux + vy)]dxdy$$

Local scale: Windowed Fourier transform

A closer look (continued)

- Following our usual plan of attack, we choose to understand the operation of the convolution via calculation of the MTF of the point spread function,

$$w(x,y)\exp[i(u_0 x + v_0 y)]$$

- Correspondingly, we consider

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(x,y)\exp[i(u_0 x + v_0 y)]\exp[-i(ux + vy)]dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(x,y)\exp\{-i[(u-u_0)x + (v-v_0)y]\}dxdy$$

Local scale: Windowed Fourier transform

A closer look (continued)

- Following our usual plan of attack, we choose to understand the operation of the convolution via calculation of the MTF of the point spread function,

$$w(x,y)\exp[i(u_0x+v_0y)]$$

- Correspondingly, we consider

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(x,y)\exp[i(u_0x+v_0y)]\exp[-i(ux+vy)]dxdy$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(x,y)\exp\{-i[(u-u_0)x+(v-v_0)y]\}dxdy$$

$$=W(u-u_0,v-v_0)$$

Where $W(u\text{-}u0,v\text{-}v0)$ is the Fourier transform of the window function, $w(x,y)$, but shifted to the "centre frequencies" $(u0,v0)$.

Local scale: Windowed Fourier transform

A closer look (continued)

- Following our usual plan of attack, we choose to understand the operation of the convolution via calculation of the MTF of the point spread function,

$$w(x,y)\exp[i(u_0x+v_0y)]$$

- Correspondingly, we consider

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(x,y)\exp[i(u_0x+v_0y)]\exp[-i(ux+vy)]dxdy$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w(x,y)\exp\{-i[(u-u_0)x+(v-v_0)y]\}dxdy$$

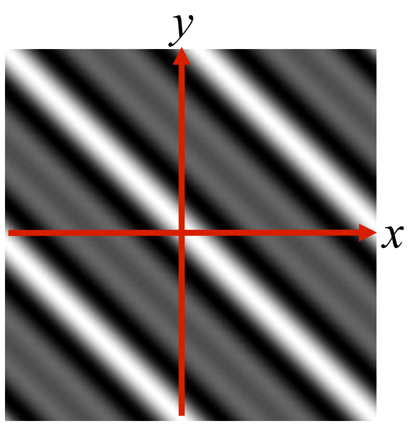
$$=W(u-u_0,v-v_0)$$

  Where $W(u\text{-}u0,v\text{-}v0)$ is the Fourier transform of the window function, $w(x,y)$, but shifted to the "centre frequencies" $(u0,v0)$.

- In words, we see that the MTF can
  - Pass the centre frequencies relatively unattenuated.
  - Suppress frequencies away from the centre so that they are attenuated according to the shape of the window function.
- We refer to such operation as that of a bandpass filter: It passes frequencies within a certain region (band) about the centre frequency.

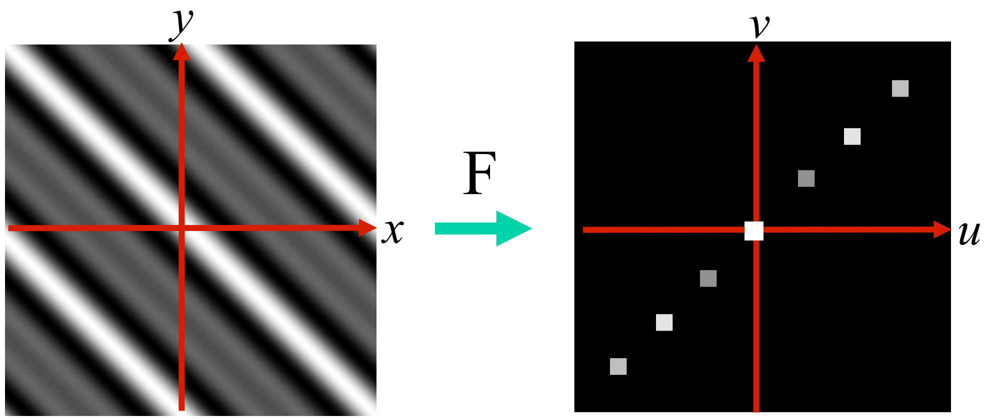
Local scale: Windowed Fourier transform

Example (pictorial)



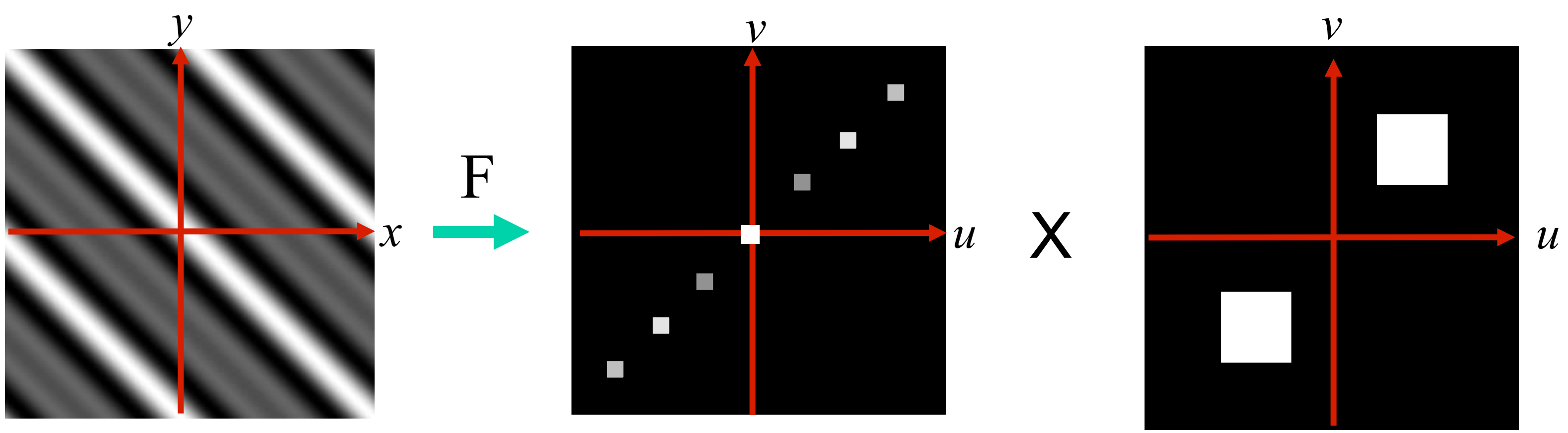
Local scale: Windowed Fourier transform

Example (pictorial)



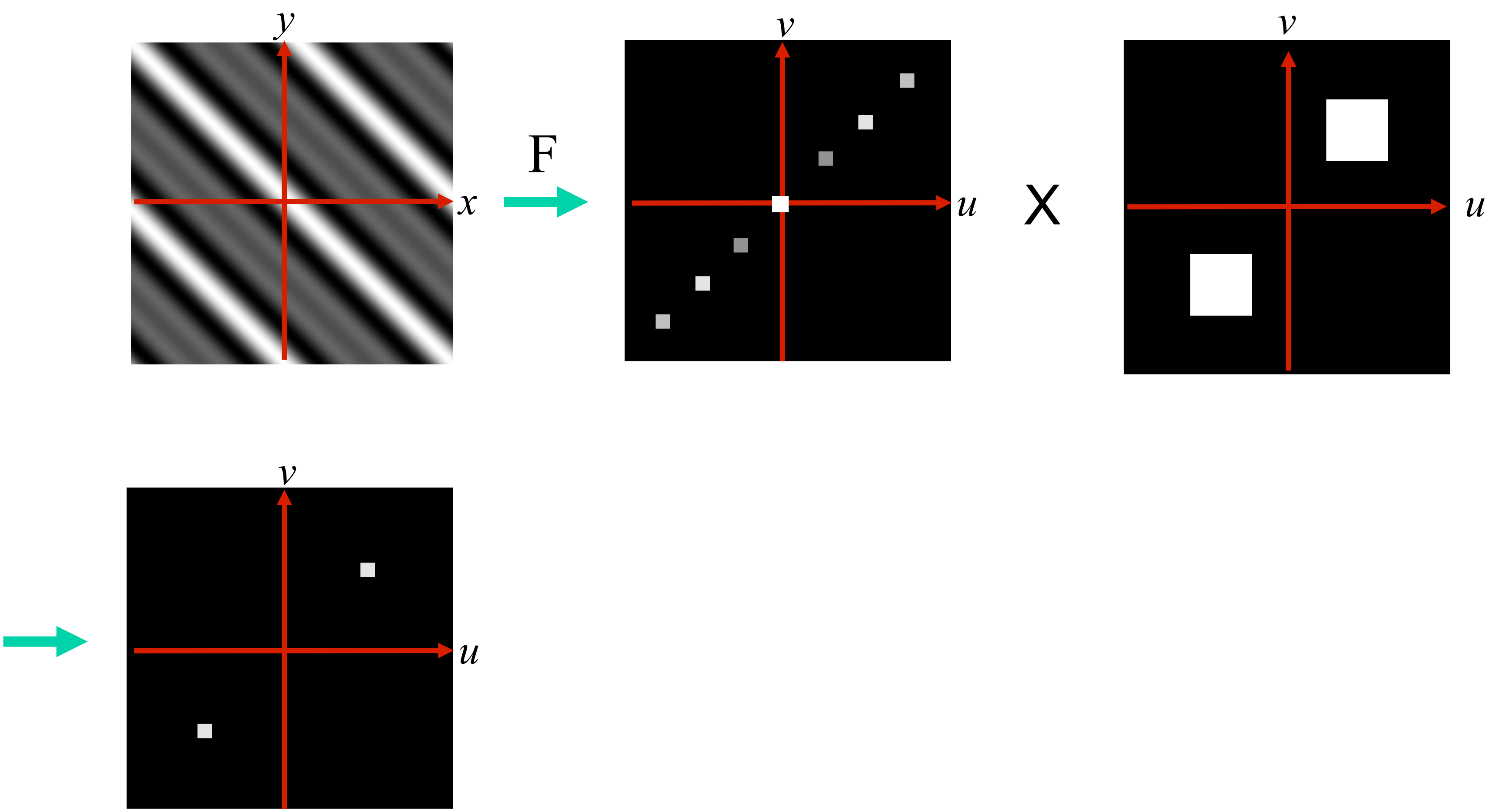
Local scale: Windowed Fourier transform

Example (pictorial)



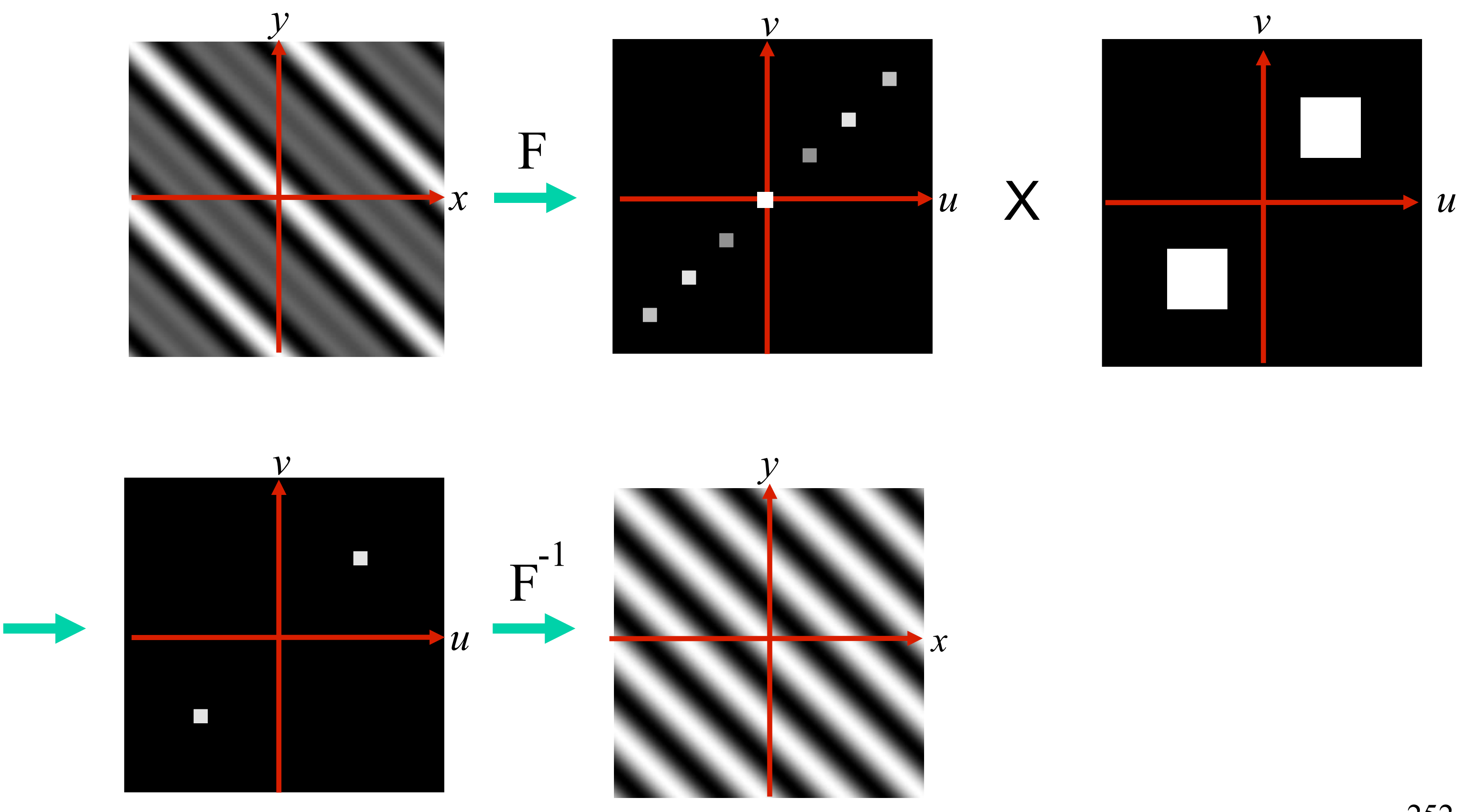
Local scale: Windowed Fourier transform

Example (pictorial)



Local scale: Windowed Fourier transform

Example (pictorial)



Local scale: Windowed Fourier transform

Example (analytic)

- Suppose that we take the window function to be that of a Gaussian

$$w(x, y) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{1}{2}\left(\frac{x^2 + y^2}{\sigma^2}\right)\right]$$

Local scale: Windowed Fourier transform

Example (analytic)

- Suppose that we take the window function to be that of a Gaussian

$$w(x,y) = \frac{1}{2\pi\sigma^2}\exp\left[-\frac{1}{2}\left(\frac{x^2+y^2}{\sigma^2}\right)\right]$$

- Then the MTF corresponding to the point spread function

$$w(x,y)\exp[i(u_0x+v_0y)]$$

is (recalling that the Fourier transform of a Gaussian is again a Gaussian with inverse standard deviation)

$$\exp\left[-\frac{1}{2}\left((u-u_0)^2+(v-v_0)^2\right)\sigma^2\right]$$

Local scale: Windowed Fourier transform

Example (analytic)

- Suppose that we take the window function to be that of a Gaussian

$$w(x,y) = \frac{1}{2\pi\sigma^2}\exp\left[-\frac{1}{2}\left(\frac{x^2+y^2}{\sigma^2}\right)\right]$$

- Then the MTF corresponding to the point spread function

$$w(x,y)\exp[i(u_0 x + v_0 y)]$$

is (recalling that the Fourier transform of a Gaussian is again a Gaussian with inverse standard deviation)

$$\exp\left[-\frac{1}{2}\left((u-u_0)^2 + (v-v_0)^2\right)\sigma^2\right]$$

- In words
  - The (centre) frequencies $(u0,v0)$ are relatively unattenuated.
  - Frequencies away from the centre are attenuated according to the Gaussian shape.

Local scale: Gabor filter

Definition

- The use of a Gaussian window in conjunction with the Fourier transform has been particularly popular.
- For example, for a given window size this choice provides a good ability to estimate precisely the local frequency content.
- Therefore, point spread functions of the form

$$\frac{1}{2\pi\sigma^2}\exp[i(u_0 x + v_0 y)]\exp\left[-\frac{1}{2}\left(\frac{x^2+y^2}{\sigma^2}\right)\right]$$

have been given a particular name, Gabor filters.

Local scale: Gabor filter

Definition

- The use of a Gaussian window in conjunction with the Fourier transform has been particularly popular.
- For example, for a given window size this choice provides a good ability to estimate precisely the local frequency content.
- Therefore, point spread functions of the form

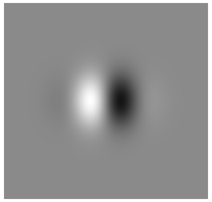
$$\frac{1}{2\pi\sigma^2}\exp[i(u_0x+v_0y)]\exp\left[-\frac{1}{2}\left(\frac{x^2+y^2}{\sigma^2}\right)\right]$$

have been given a particular name, Gabor filters.

Remark

- In practice, such filters are applied by splitting them into their cosinusoidal and sinusoidal components

$$\frac{1}{2\pi\sigma^2}\cos(u_0x+v_0y)\exp\left[-\frac{1}{2}\left(\frac{x^2+y^2}{\sigma^2}\right)\right]$$



$$\frac{1}{2\pi\sigma^2}\sin(u_0x+v_0y)\exp\left[-\frac{1}{2}\left(\frac{x^2+y^2}{\sigma^2}\right)\right]$$

Local scale: Quadrature filters

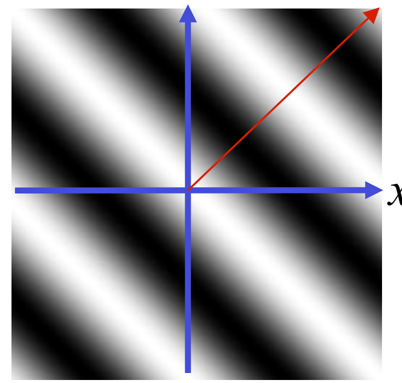
Some more terminology

- Gabor filters are examples of quadrature filters.
- That is, the sinusoidal and cosinusoidal components have the same MTF except that they are shifted in phase by $\pi / 2$.
- Technically, they are related by the so called Hilbert transform.
- Just in case anybody asks…

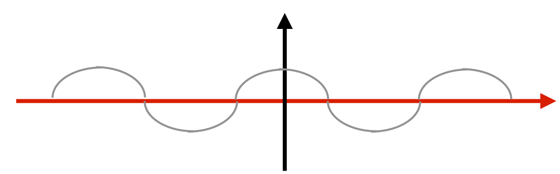
Local scale & orientation: Relating space & frequency

Recall

- For given u and v in the frequency domain, there corresponds a (cos)sinusoidal waveforms, $\cos(ux+vy)$ and $\sin(ux+vy)$, at a particular orientation and periodicity in the spatial domain.



The maxima and minima of the cosinusoids lie along parallel equidistant lines $ux + vy = k\pi$ for k an integer.



Cross sections orthogonal to the ridges show a sinusoidal profile with wavelength $2\pi \big/ \sqrt{u^2 + v^2}$

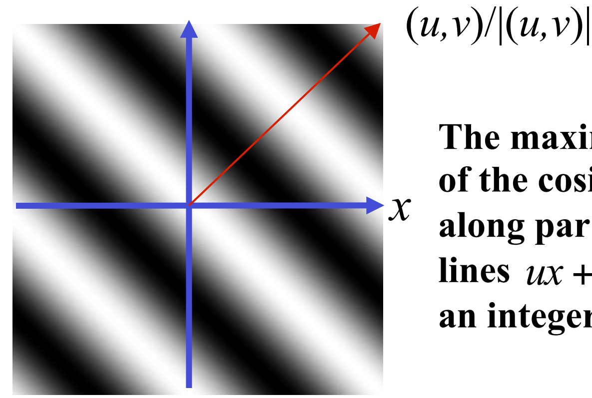
Local scale & orientation: Relating space and frequency

Recall

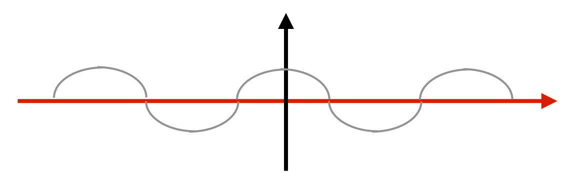
- For given u and v in the frequency domain, there corresponds a (cos)sinusoidal waveforms, $\cos(ux+vy)$ and $\sin(ux+vy)$, at a particular orientation and periodicity in the spatial domain.

Now

- Through application of the filters that we have just constructed, we can parse the image information according to its local structural components
  - Scale: magnitude of the (center) frequency $|u0,v0|$
  - Orientation: direction of sinusoid $(u0,v0)/|u0,v0|$.



The maxima and minima of the cosinusoids lie along parallel equidistant lines $ux + vy = k\pi$ **for** k **an integer.**



Cross sections orthogonal to the ridges show a sinusoidal profile with wavelength $2\pi / \sqrt{u^2 + v^2}$

Local scale & orientation: Scale

Recall

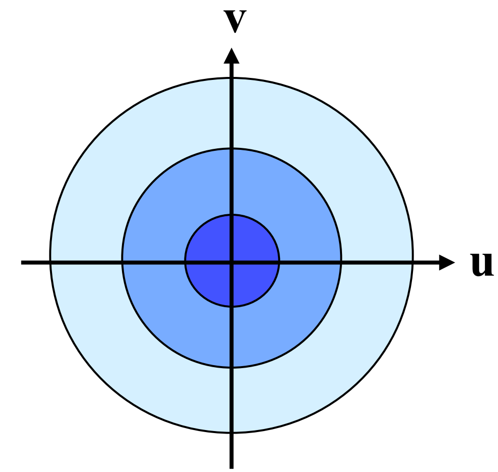
- For given u and v in the frequency domain, there corresponds a (cos)sinusoidal waveforms, $\cos(ux+vy)$ and $\sin(ux+vy)$, at a particular orientation and periodicity in the spatial domain.

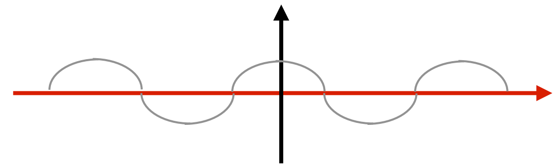
Now

- Through application of the filters that we have just constructed, we can parse the image information according to its local structural components
  - Scale: magnitude of the (center) frequency $|u0,v0|$
  - Orientation: direction of sinusoid $(u0,v0)/|u0,v0|$.

Consider

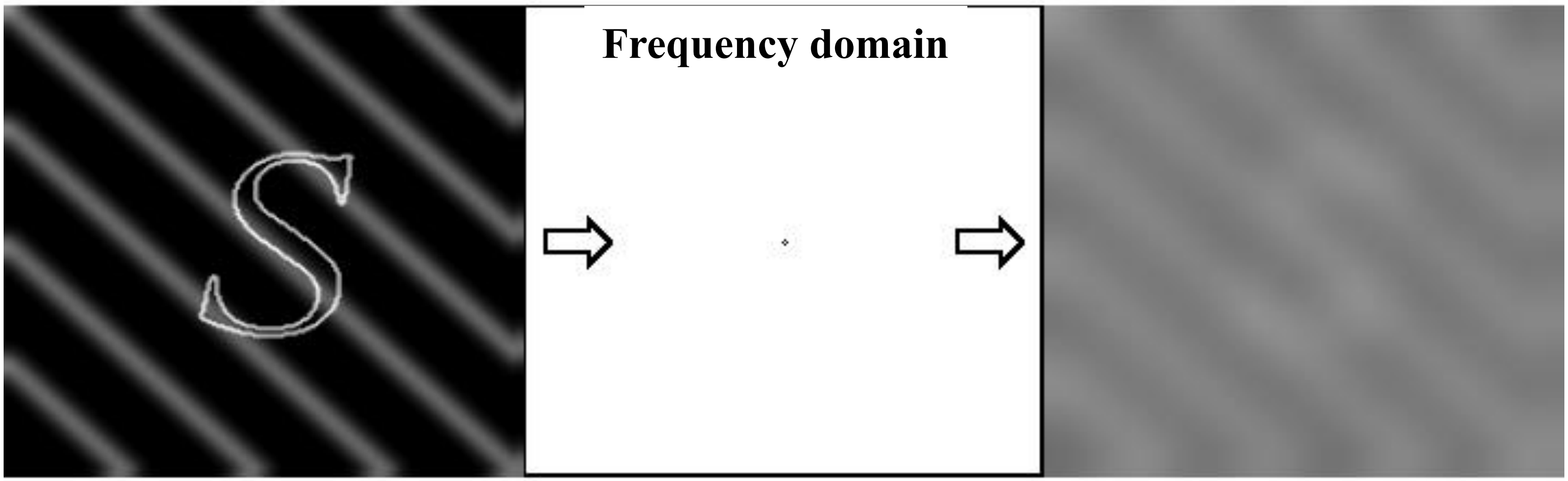
- For the sake of illustration, let's think about just the scale component.
- Here for a given $(u0,v0)$ we consider an annulus of frequencies.



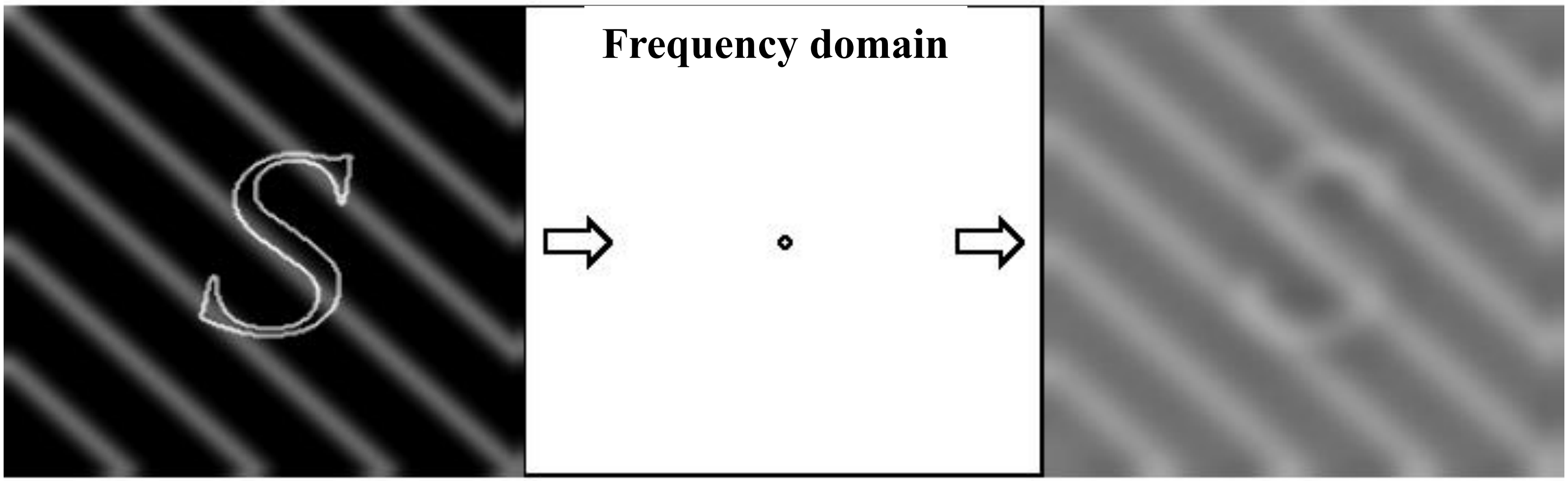


Cross sections orthogonal to the ridges show a sinusoidal profile with wavelength $2\pi / \sqrt{u^2 + v^2}$

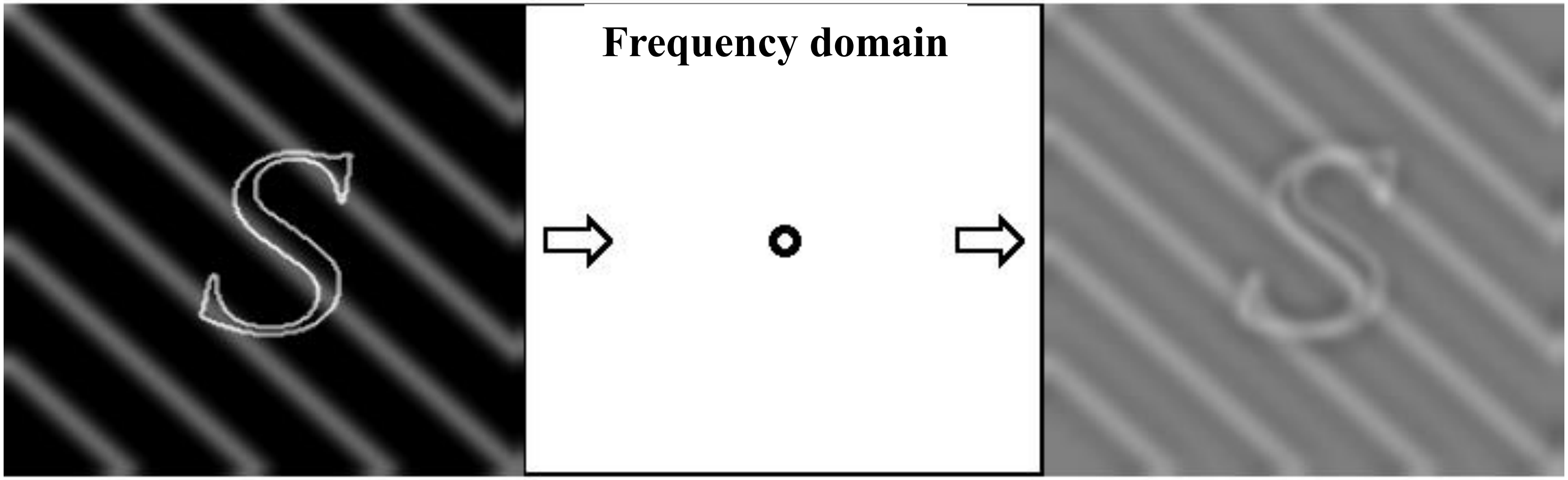
Local scale: Scale selection



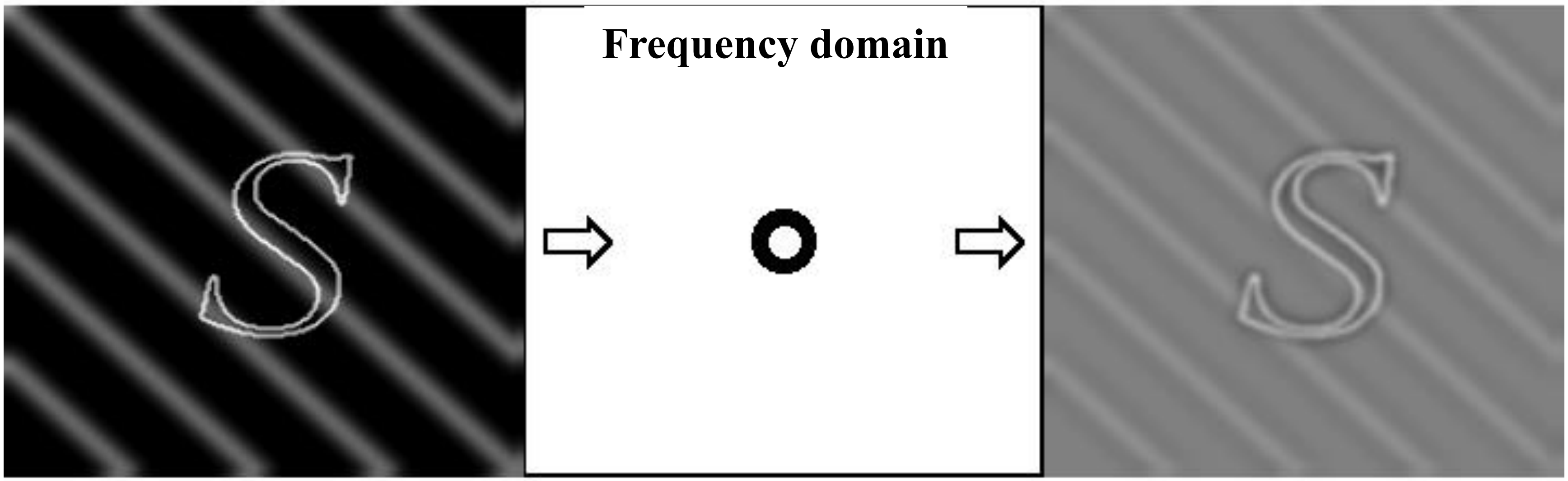
Local scale: Scale selection



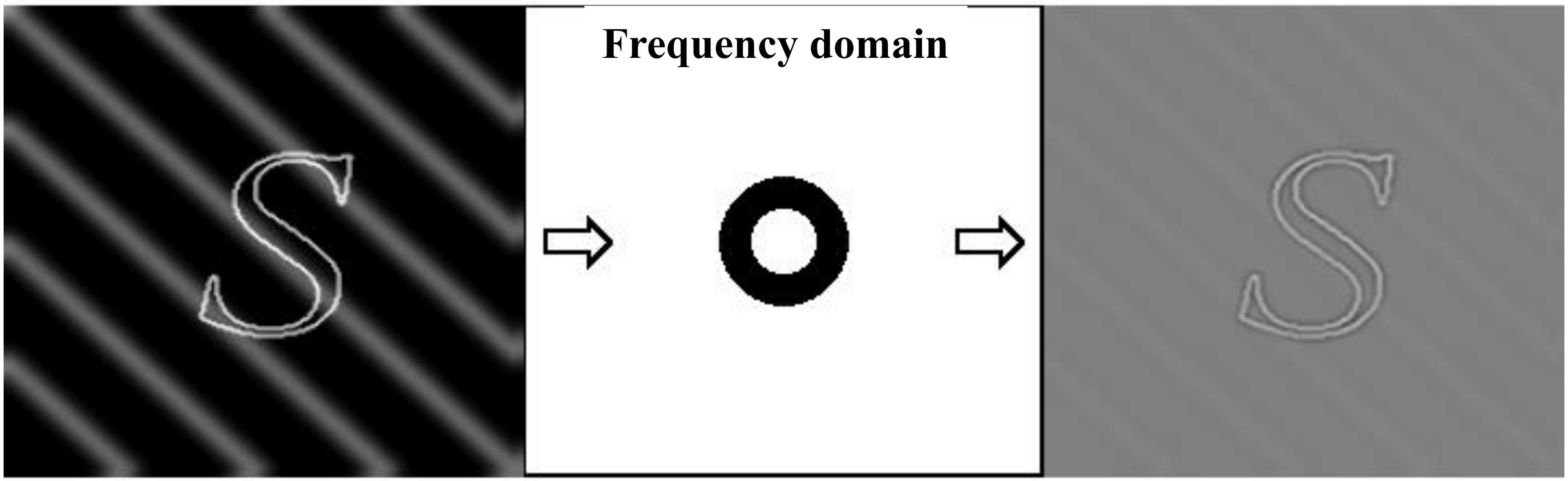
Local scale: Scale selection



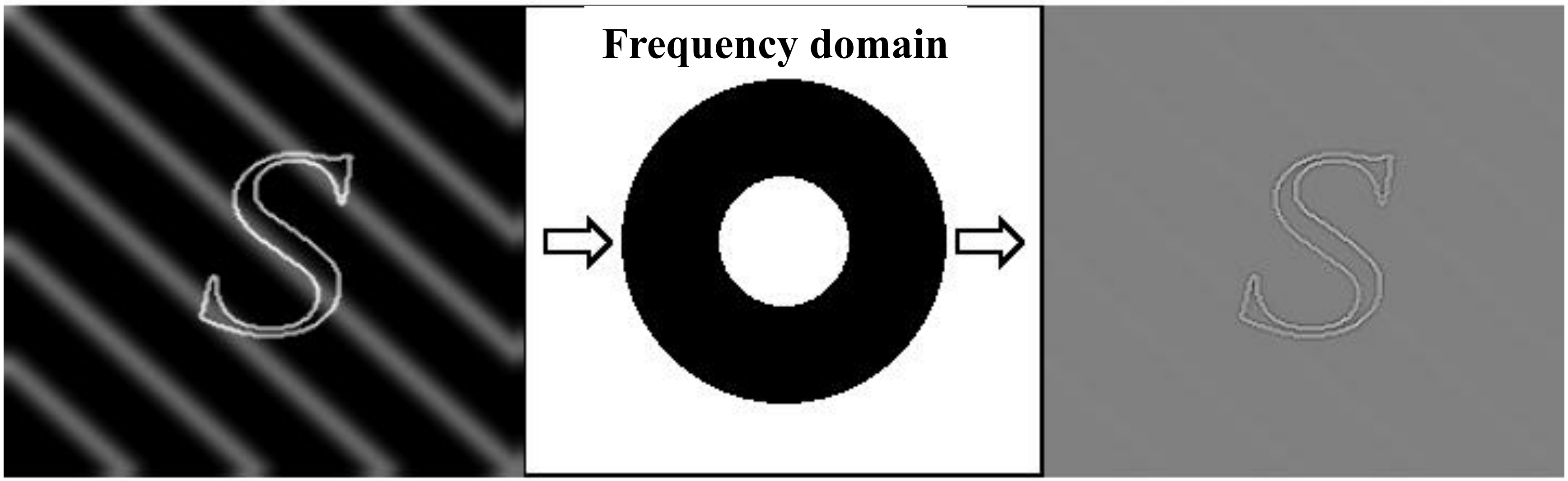
Local scale: Scale selection



Local scale: Scale selection



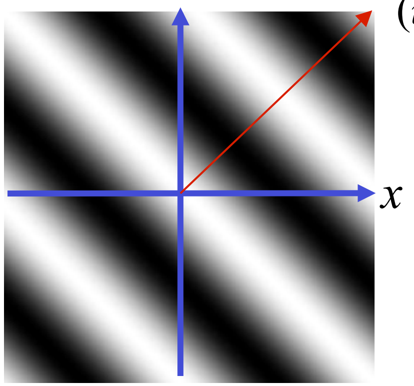
Local scale: Scale selection



Local scale & orientation: Orientation

Recall

- For given u and v in the frequency domain, there corresponds a (cos)sinusoidal waveforms, $\cos(ux+vy)$ and $\sin(ux+vy)$, at a particular orientation and periodicity in the spatial domain.



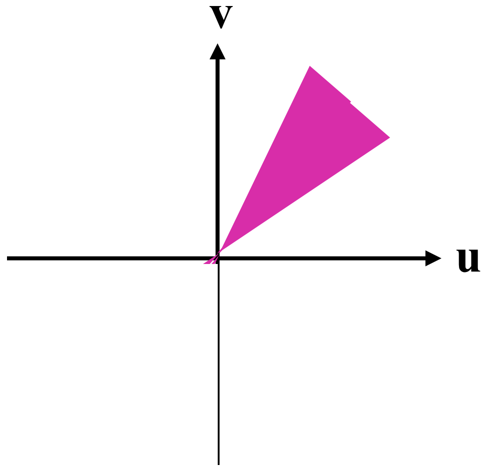
The maxima and minima of the cosinusoids lie along parallel equidistant lines $ux + vy = k\pi$ for k an integer.

Now

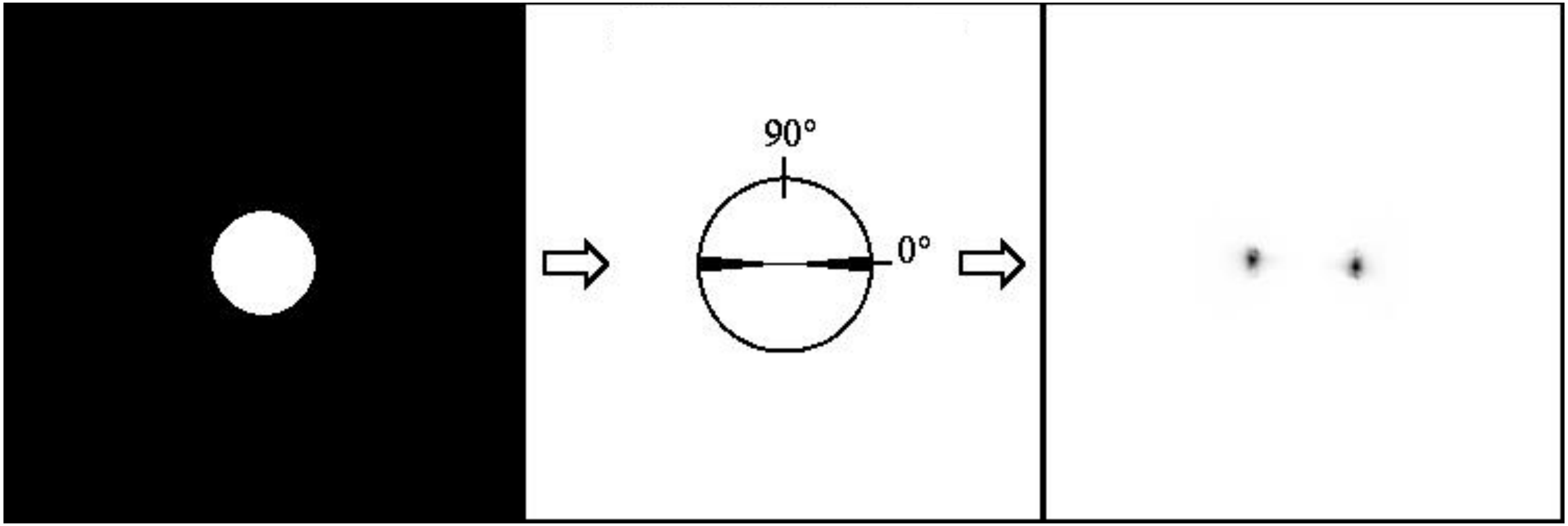
- Through application of the filters that we have just constructed, we can parse the image information according to its local structural components
  - Scale: magnitude of the (center) frequency $|u0,v0|$
  - Orientation: direction of sinusoid $(u0,v0)/|u0,v0|$

Consider

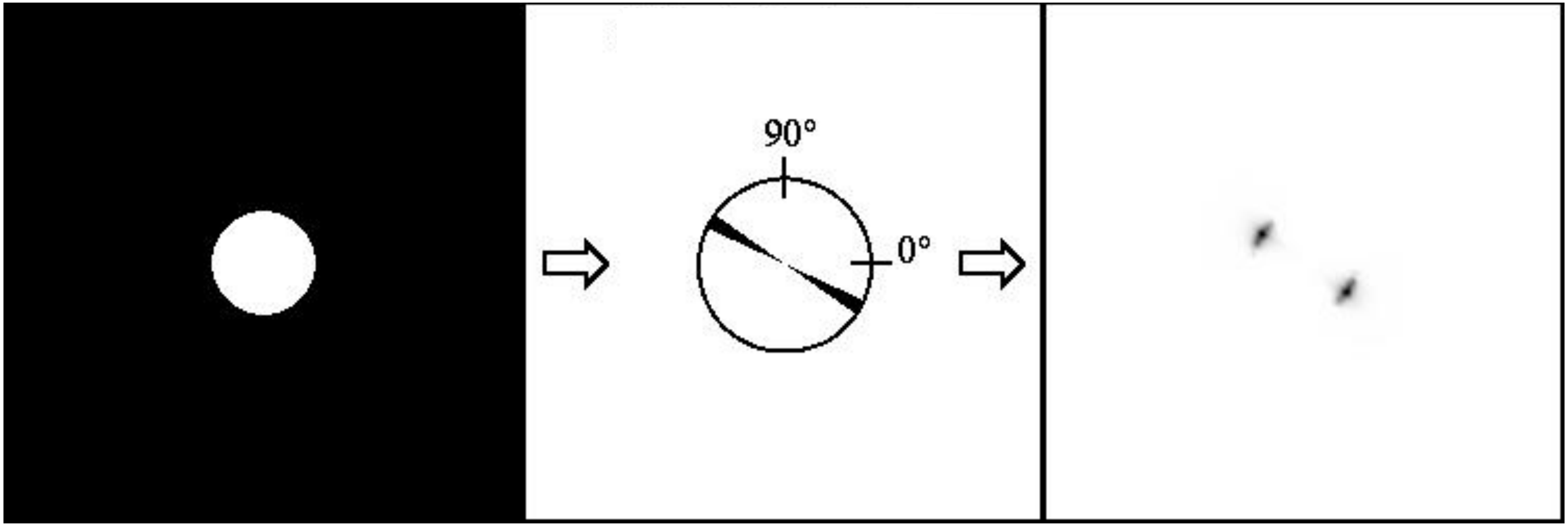
- For the sake of illustration, let's think about just the orientation component.
- Here for a given $(u0,v0)$ we consider a slice of frequencies.



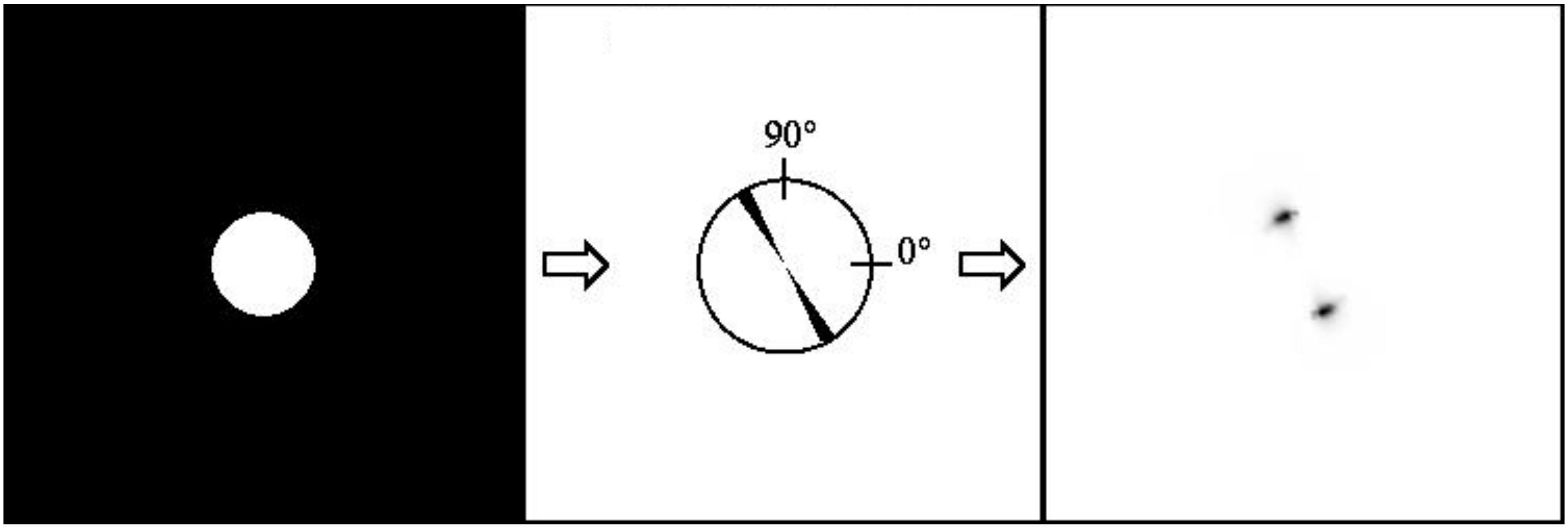
Local orientation: Orientation selection



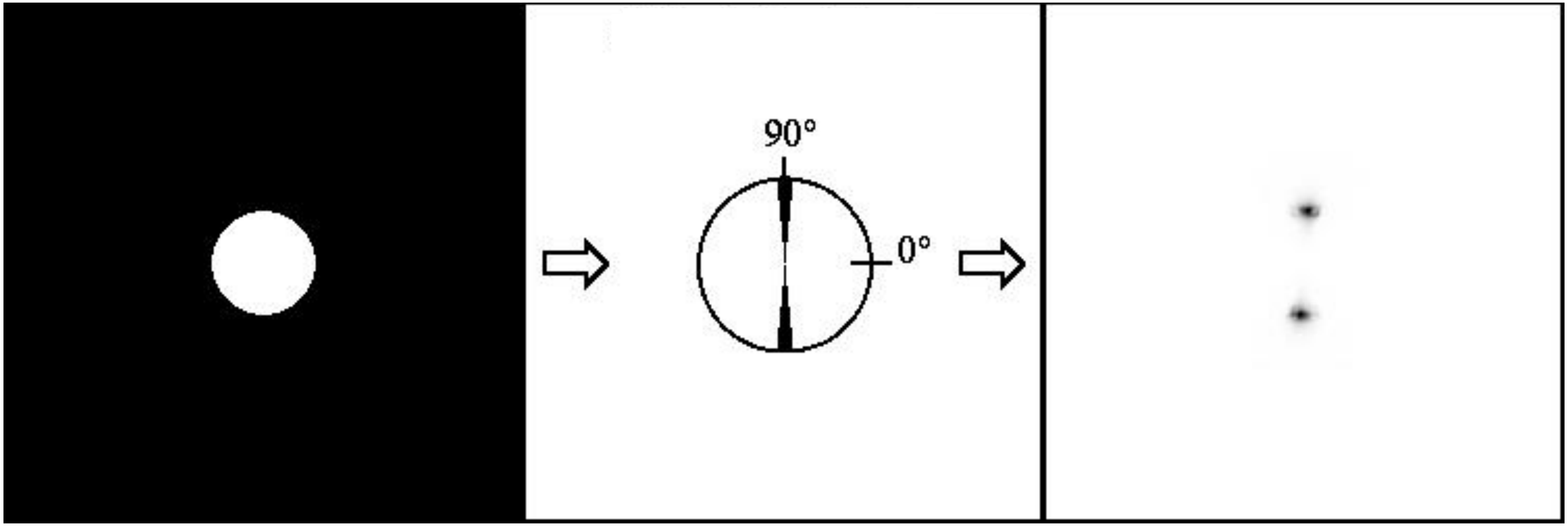
Local orientation: Orientation selection



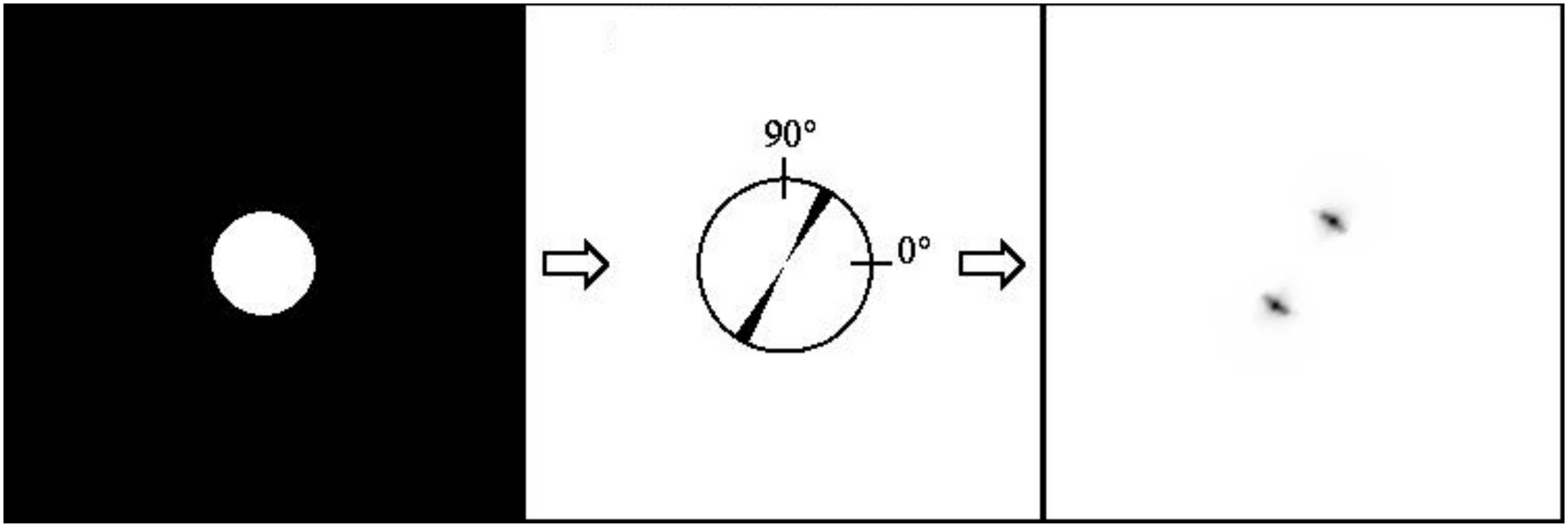
Local orientation: Orientation selection



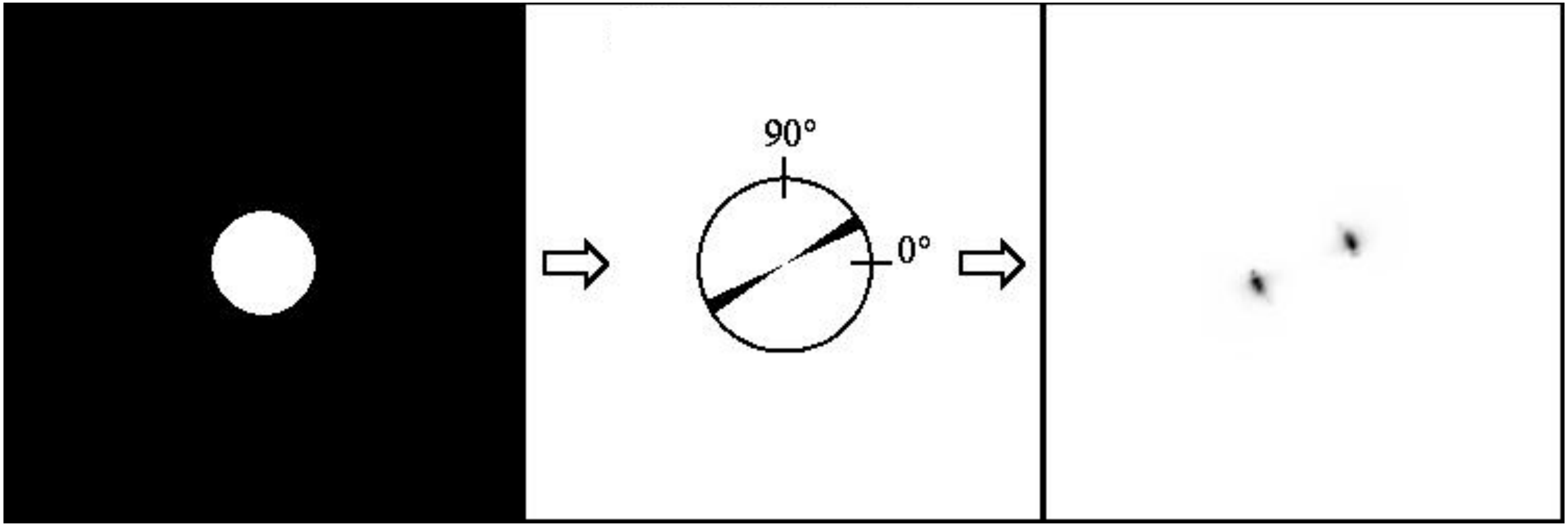
Local orientation: Orientation selection



Local orientation: Orientation selection

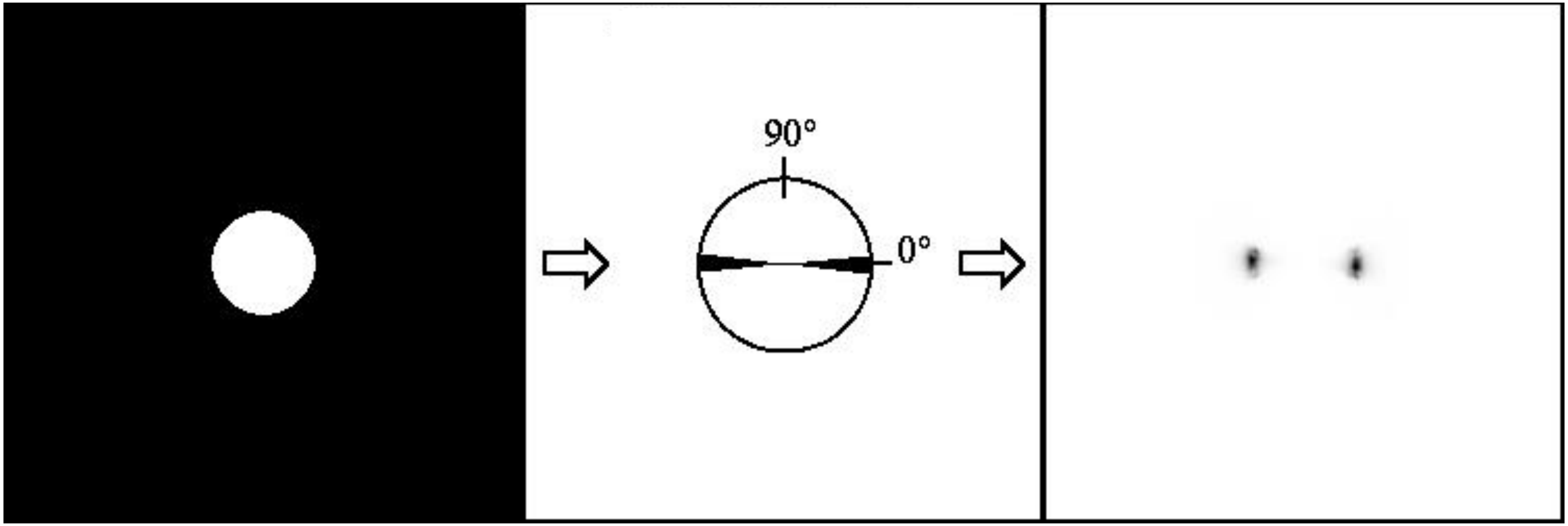


Local orientation: Orientation selection

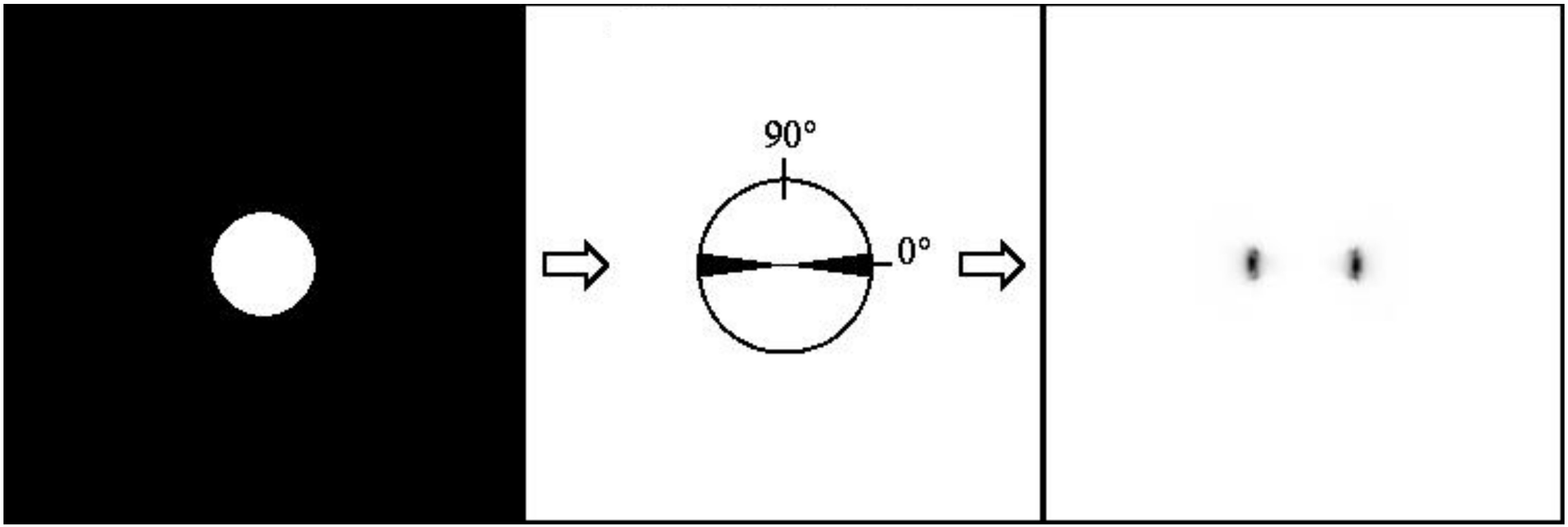


Local orientation: Bandwidth selection

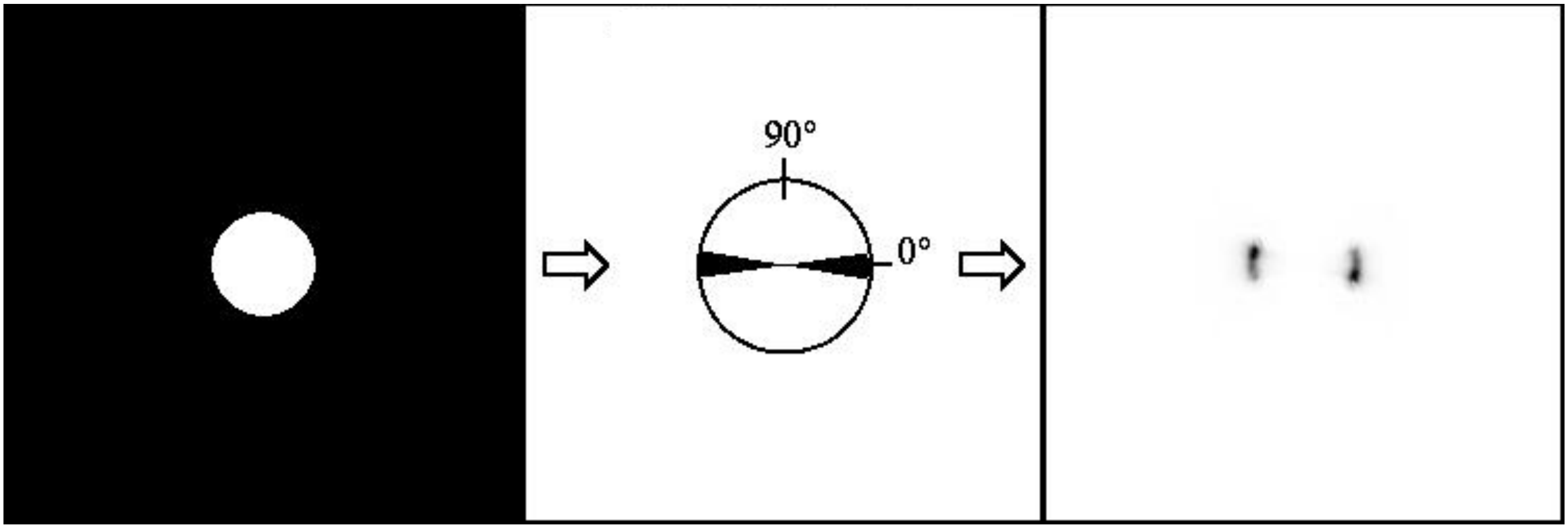
Consider the effects of keeping orientation constant, but varying bandwidth.



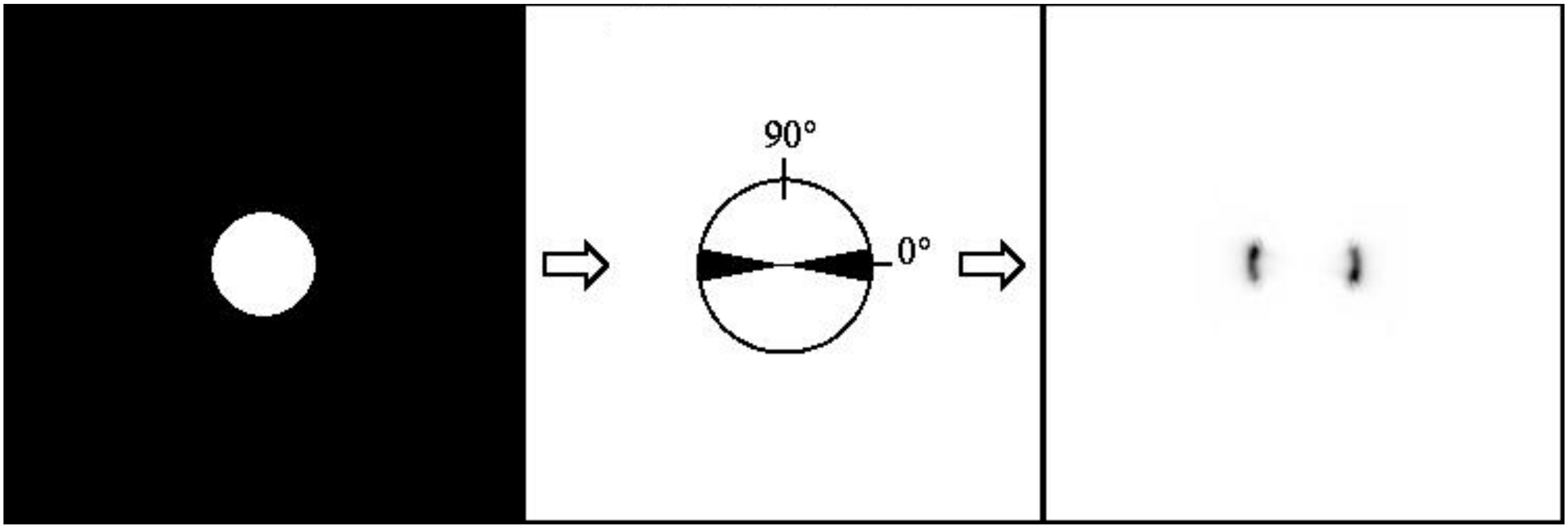
Local orientation: Bandwidth selection



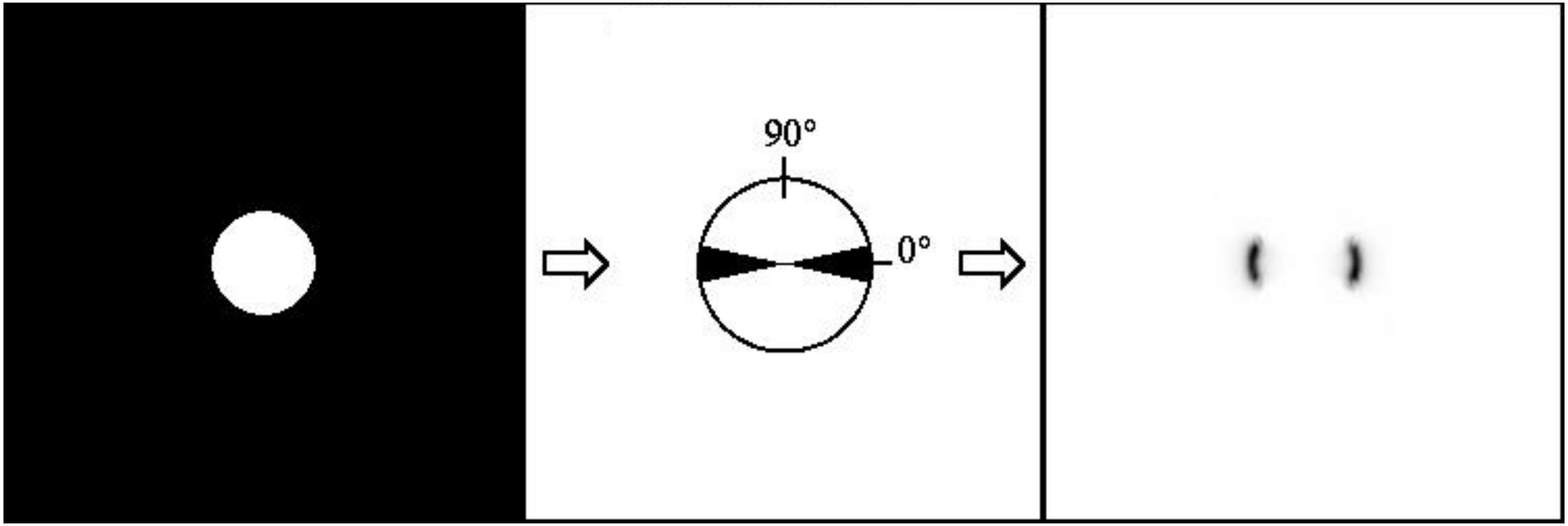
Local orientation: Bandwidth selection



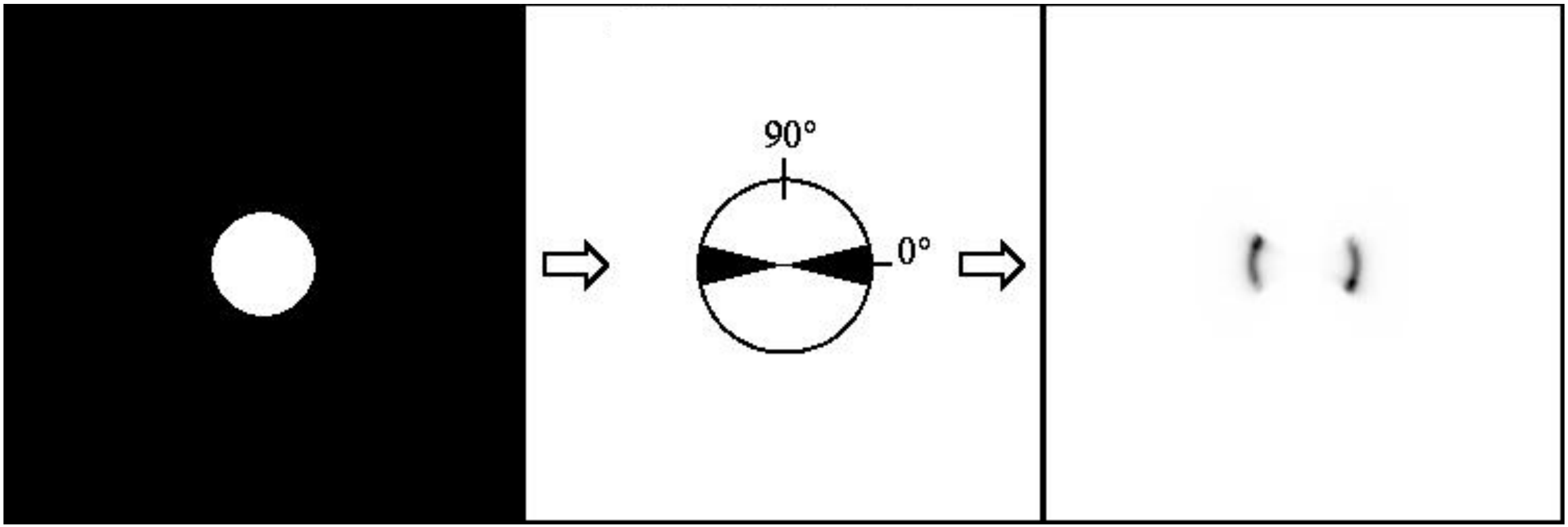
Local orientation: Bandwidth selection



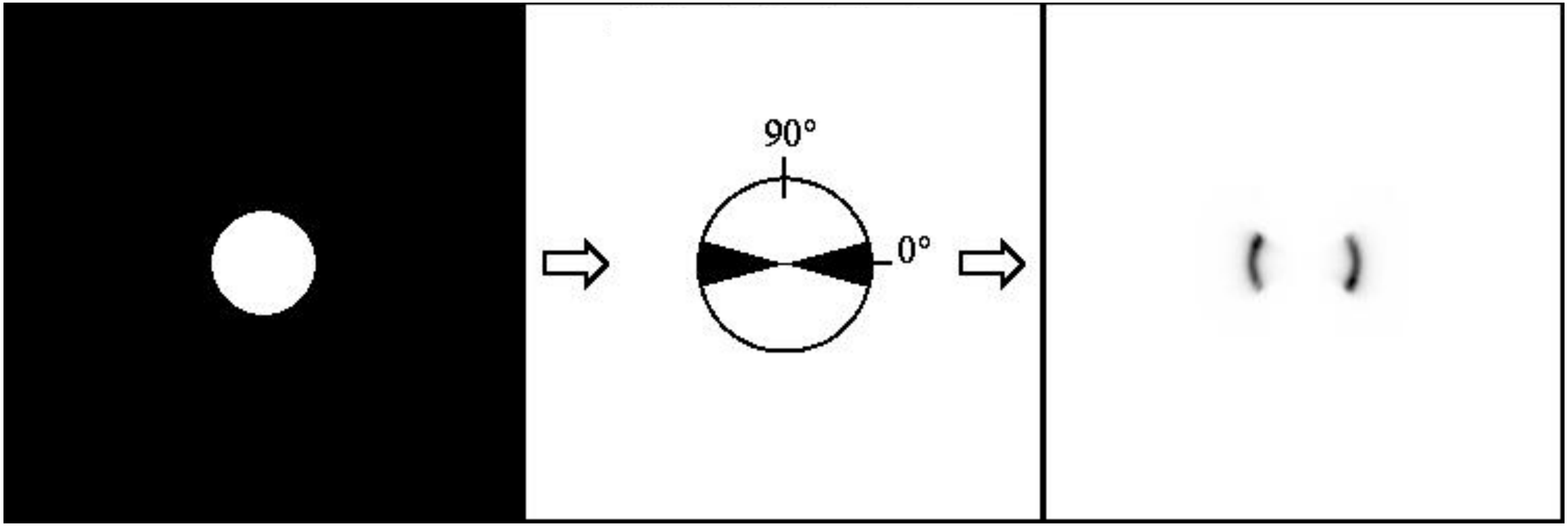
Local orientation: Bandwidth selection



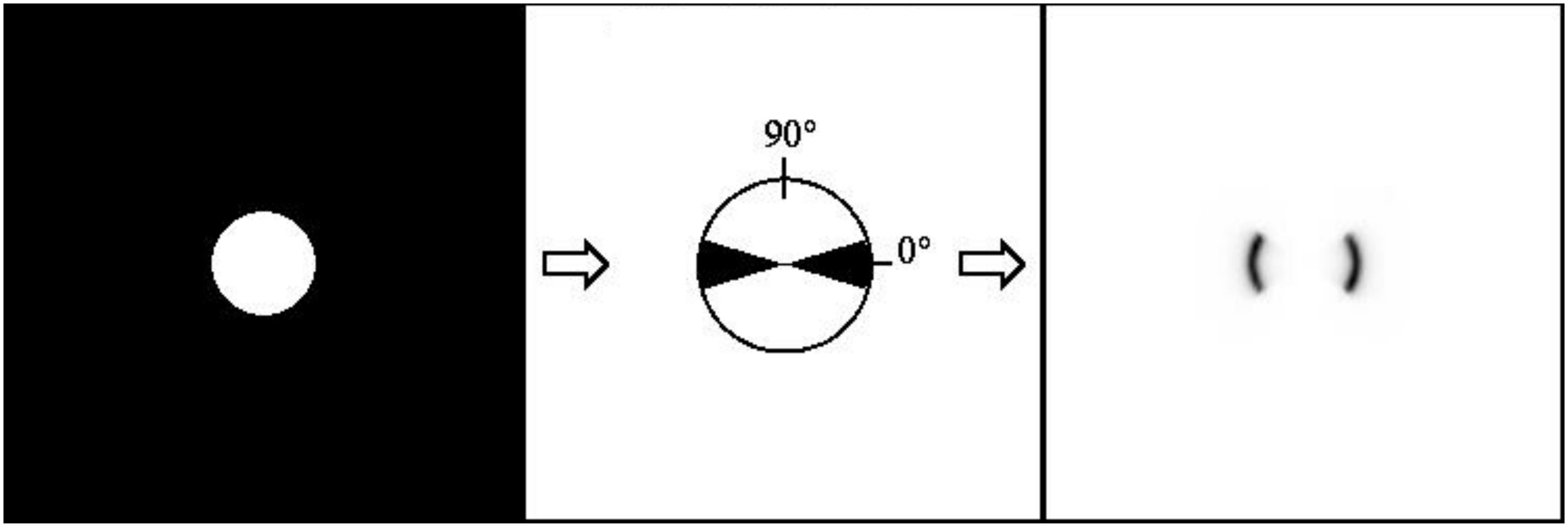
Local orientation: Bandwidth selection



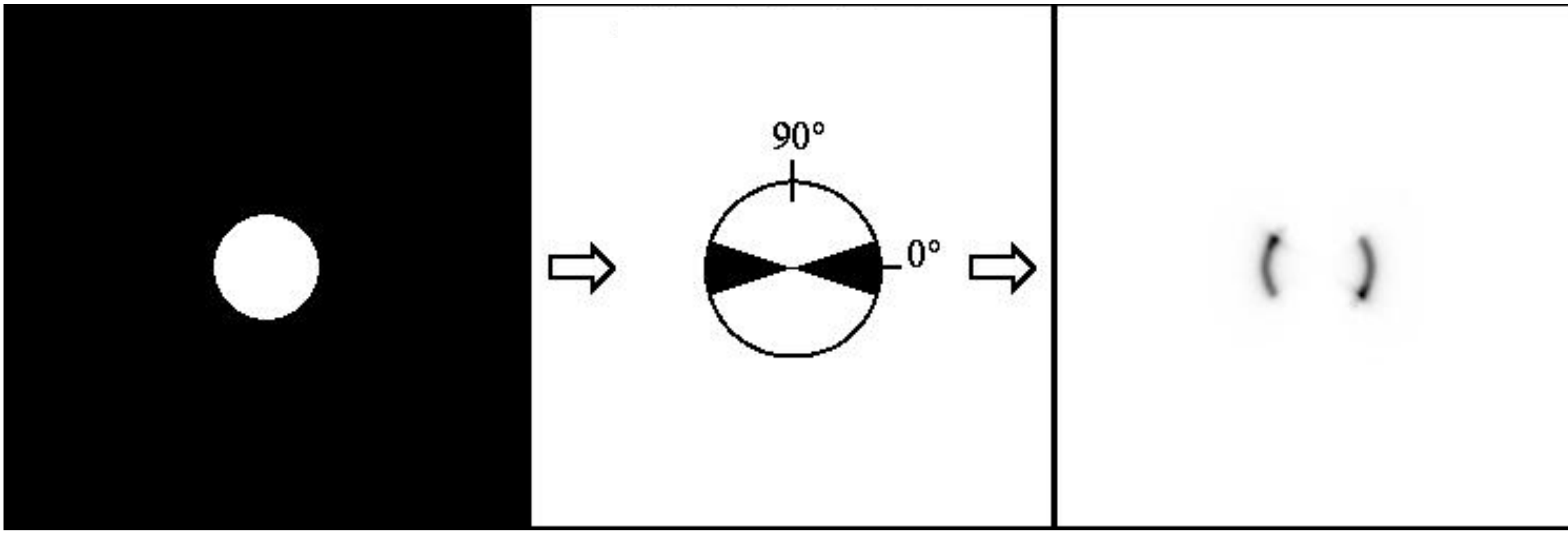
Local orientation: Bandwidth selection



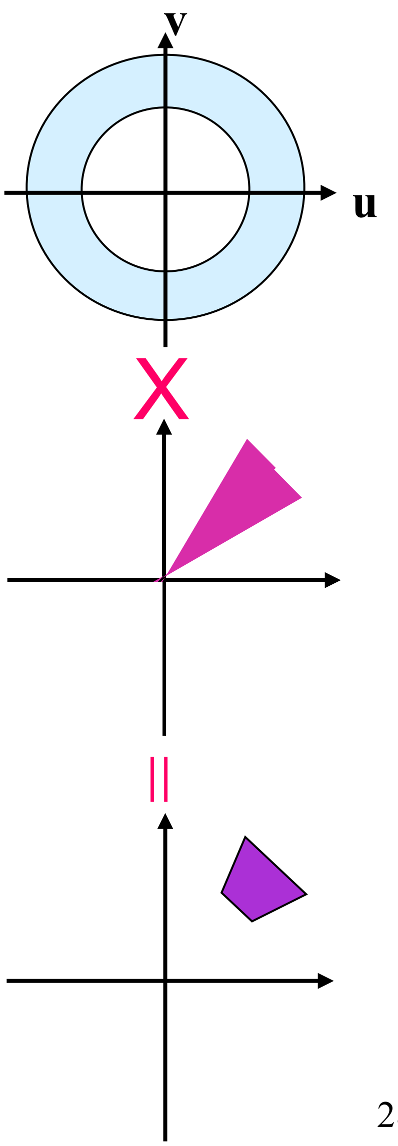
Local orientation: Bandwidth selection



Local orientation: Bandwidth selection



Local scale & orientation: Combined analysis



Recall

- For given u and v in the frequency domain, there corresponds a (cos)sinusoidal waveforms, $\cos(ux+vy)$ and $\sin(ux+vy)$, at a particular orientation and periodicity in the spatial domain.

Now

- Through application of the filters that we have just constructed, we can parse the image information according to its local structural components
  - Scale: magnitude of the (center) frequency $|u0,v0|$
  - Orientation: direction of sinusoid $(u0,v0)/|u0,v0|$

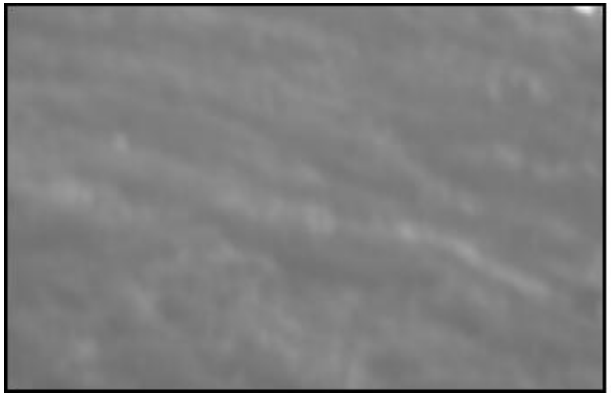
Consider

- Combine selection for both scale and orientation.
- Here for a given $(u0,v0)$ we consider a wedge of frequencies.

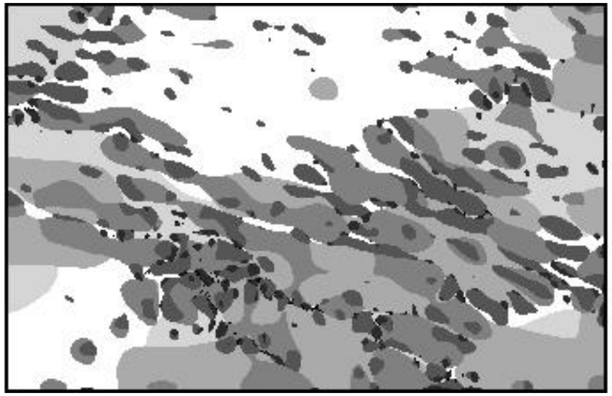
Image representation: Local scale x orientation analysis

Example

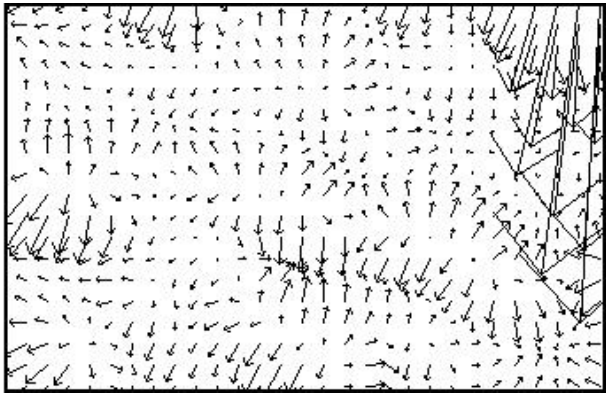
• Seek to guide subsequent processing by pointing way to locally characteristic/dominant image structure.
• Decompose an image according to local scale and orientation via application of bandpass filters.
• Select locally dominant scale and orientation via scanning the resulting representation for strongest responses.



Source image (natural terrain)



Locally dominant scale (darker intensity for finer scale)

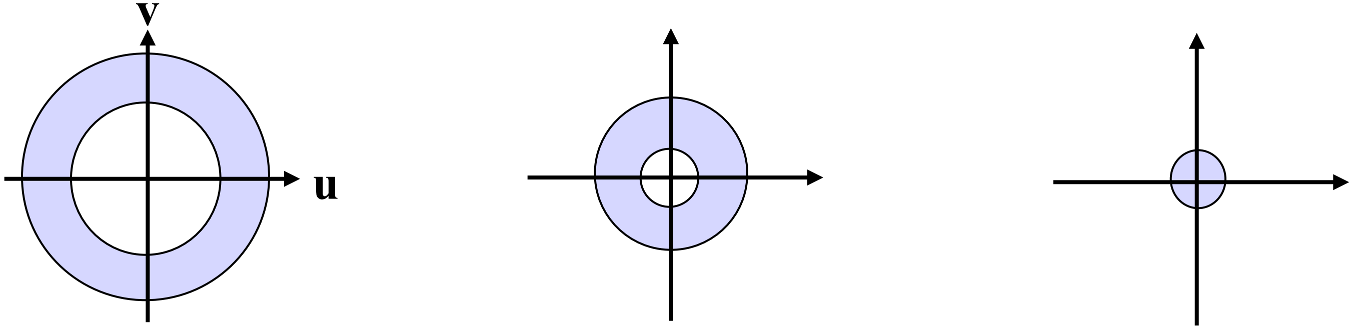


Locally dominant orientation (shown as normal vector)

Local scale: Lowpass filters

Intuition

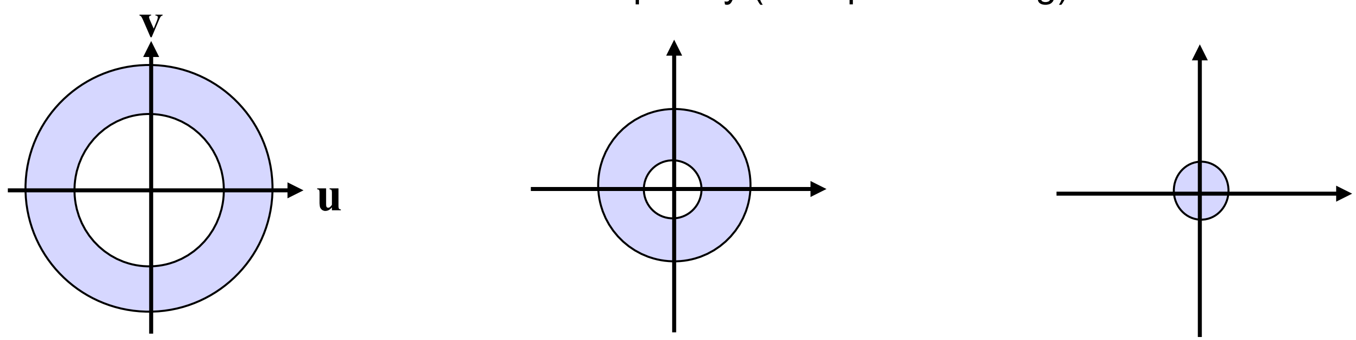
- We have considered representation of an image by decomposing it according the frequency content within a band about a central frequency (bandpass filtering).



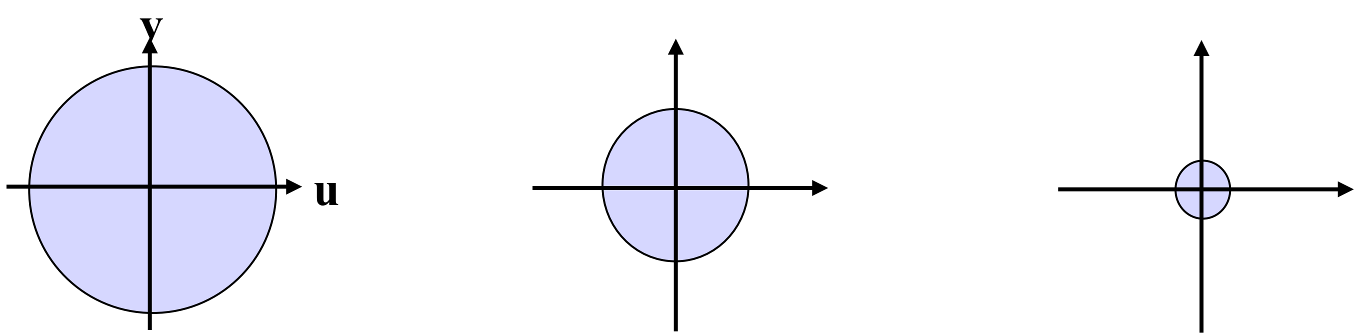
Local scale: Lowpass filters

Intuition

- We have considered representation of an image by decomposing it according the frequency content within a band about a central frequency (bandpass filtering).



- A complimentary approach is to represent an image by successively removing its higher frequency components.



Local scale: Lowpass filters

Formalization

- Apparently we seek a MTF that can be manipulated so as to cover variable portions of the frequency domain, centered about the origin.
- As an example, we can make use of a Gaussian window function centered about the origin in the frequency domain.

$$\exp\left[-\frac{1}{2}\left(u^2+v^2\right)\sigma^2\right]$$

  – When a small value for sigma is used, much of the frequency information will be passed relatively unattenutated.
  – As larger values for sigma are used only the lower frequencies are passed without severe attenuation.
  – We refer to such filters as lowpass filters.

Local scale: Lowpass filters

Formalization

- Apparently we seek a MTF that can be manipulated so as to cover variable portions of the frequency domain, centered about the origin.
- As an example, we can make use of a Gaussian window function centered about the origin in the frequency domain.

$$\exp\left[-\frac{1}{2}\left(u^2+v^2\right)\sigma^2\right]$$

  - When a small value for sigma is used, much of the frequency information will be passes relatively unattenutated.
  - As larger values for sigma are used only the lower frequencies are passed without severe attenuation.
  - We refer to such filters as lowpass filters.
- Recalling that the inverse Fourier transform is again a Gaussian (with inverse standard deviation), the point spread function in the frequency domain must be have the form

$$\frac{1}{2\pi\sigma^2}\exp\left[-\frac{1}{2}\left(\frac{x^2+y^2}{\sigma^2}\right)\right]$$

  - When small standard deviations are used, much of the frequency information will be passed relatively unattended.
  - When large standard deviations are used, only the lower frequencies are passed without severe attenuation.

Local scale: Lowpass example

Local scale: Scale space

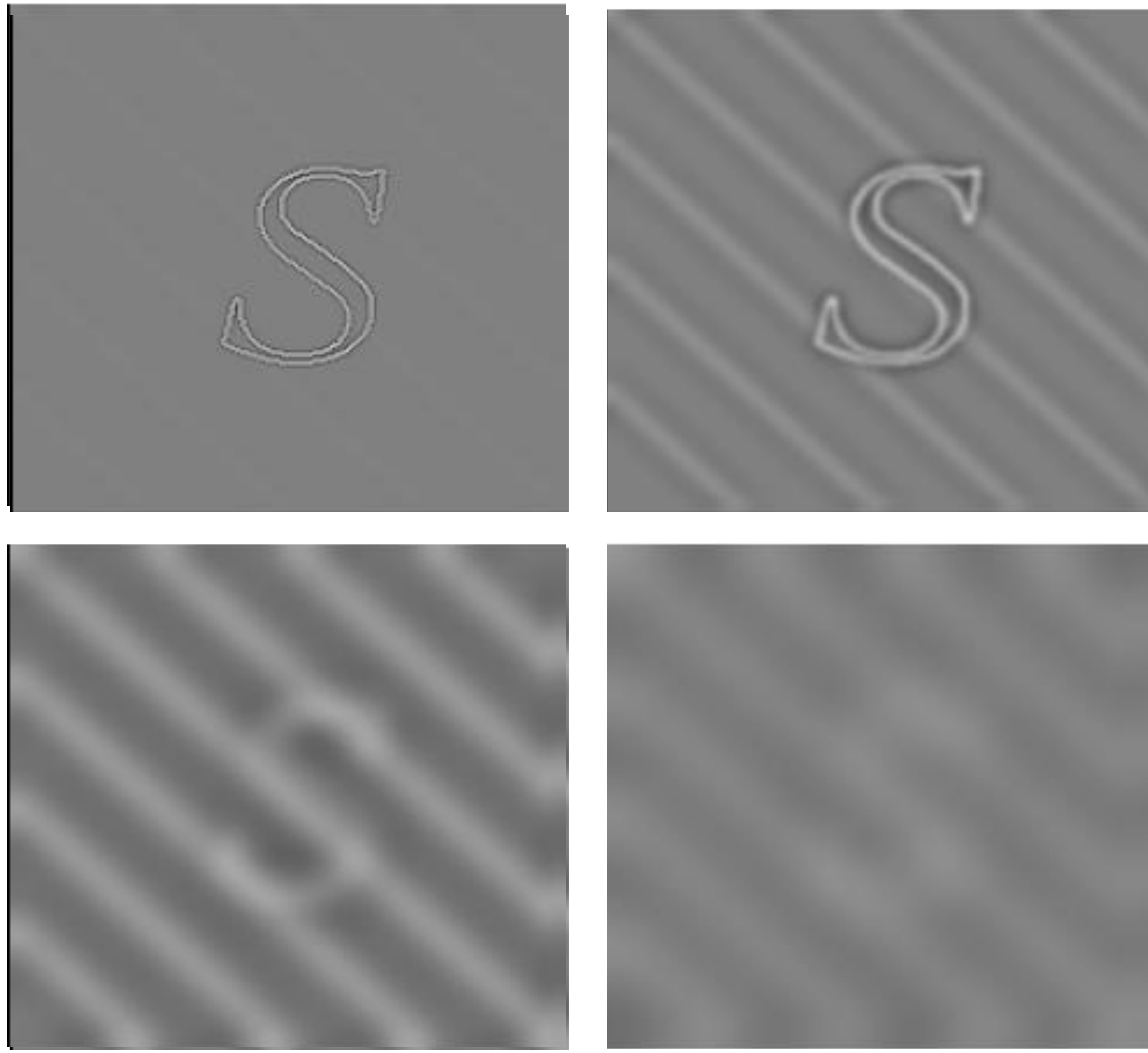
Concept: We add an additional axis to our image representation where scale (lowpass or bandpass) is the new dimension. (Here we show lowpass.)

Advantage: Provides a principled parsing of the local image structure.

Local scale: Scale space

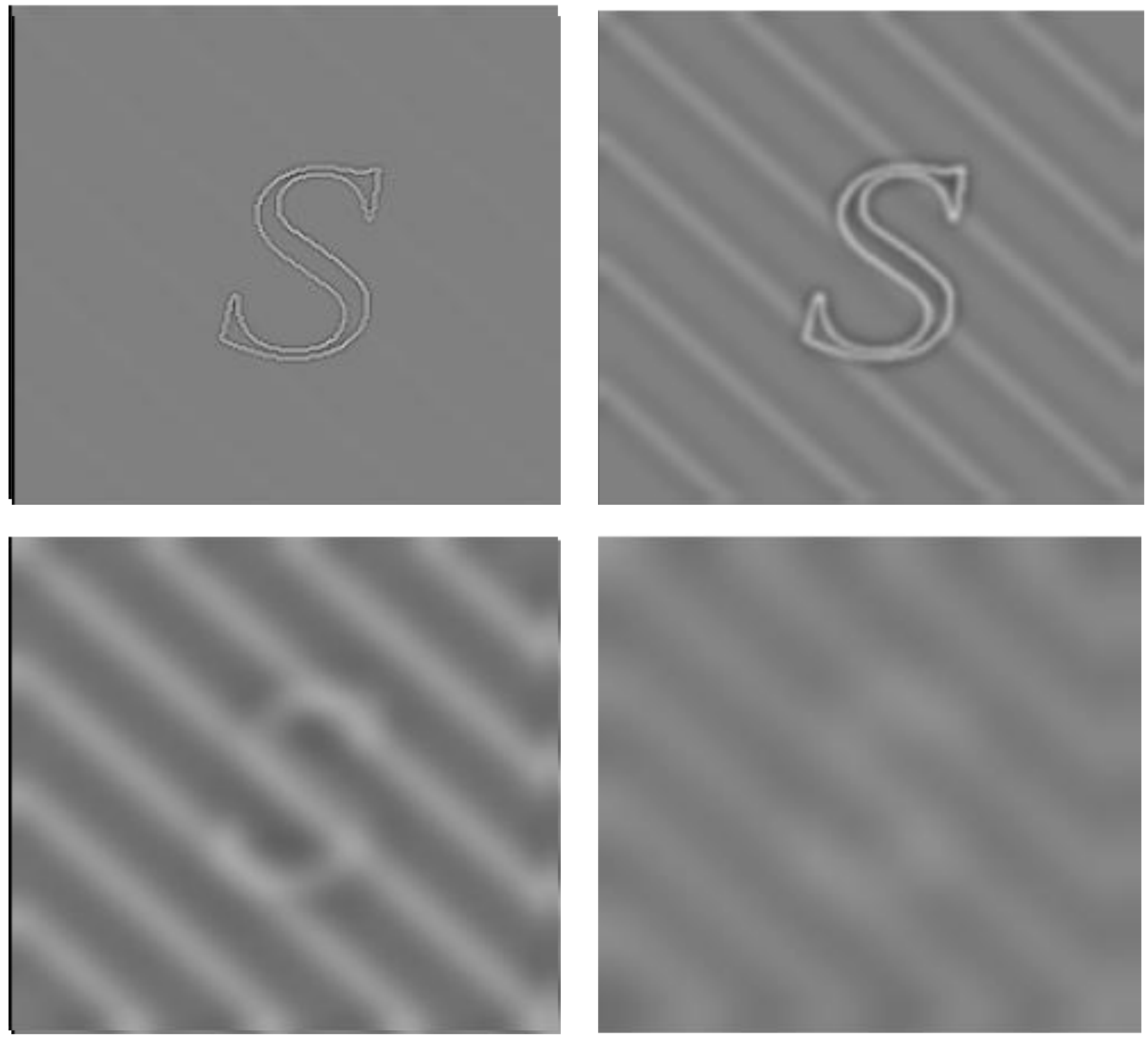
Concept: We add an additional axis to our image representation where scale (lowpass or bandpass) is the new dimension. (Here we show bandpass.)

Advantage: Provides a principled parsing of the local image structure.



Local scale: Scale space

Concept: We add an additional axis to our image representation where scale (lowpass or bandpass) is the new dimension. (Here we show bandpass.)

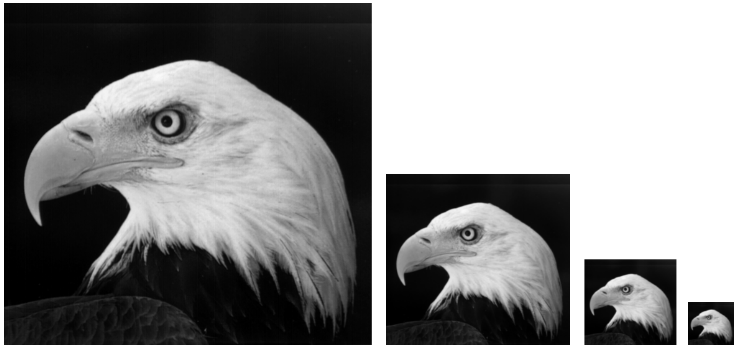


Disadvantage: A potential explosion of storage requirements.

Local scale: Pyramids

Intuition

- Scale space parses information according to spatial frequency content.
- By the sampling theorem, lower spatial frequencies can be captured with coarser spatial sampling.
- It should be possible to “subsample” the scale space components that correspond to lower frequencies.

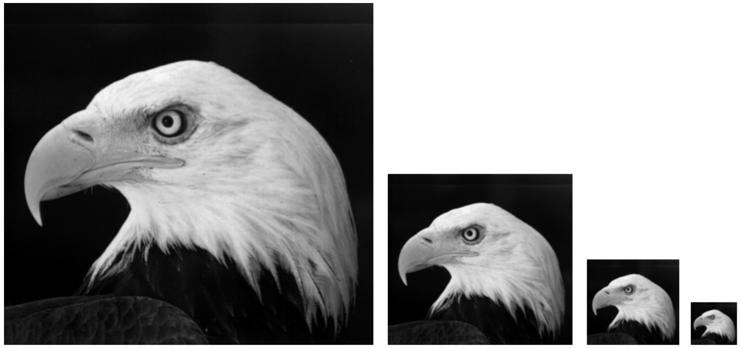


Local scale: Pyramids

Intuition

- Scale space parses information according to spatial frequency content.
- By the sampling theorem, lower spatial frequencies can be captured with coarser spatial sampling.
- It should be possible to “subsample” the scale space components that correspond to lower frequencies.

Gaussian pyramid



Formalization

- Lowpass Gaussian pyramids can be constructed by successively lowpass filtering the image and taking every other row and column (taking care that that the highest passed frequency can be properly captured).

Local scale: Pyramids

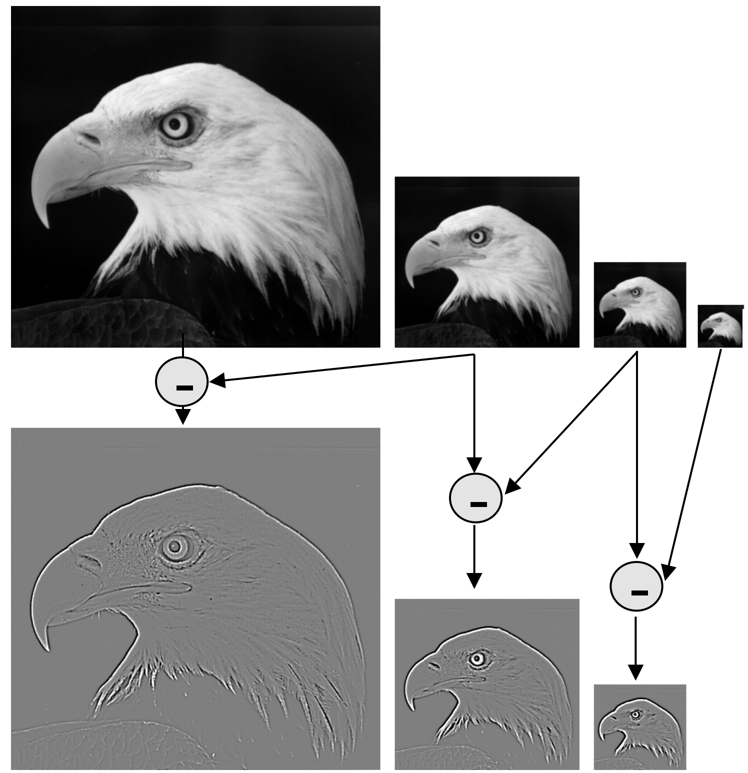
Intuition

- Scale space parses information according to spatial frequency content.
- By the sampling theorem, lower spatial frequencies can be captured with coarser spatial sampling.
- It should be possible to "subsample" the scale space components that correspond to lower frequencies.

Formalization

- Lowpass Gaussian pyramids can be constructed by successively lowpass filtering the image and taking every other row and column (taking care that that the highest passed frequency can be properly captured).
- Bandpass Laplacian pyramids can be constructed by taking the pointwise difference of successive levels in the lowpass pyramid.

Gaussian pyramid



Laplacian pyramid

Local scale: Pyramids

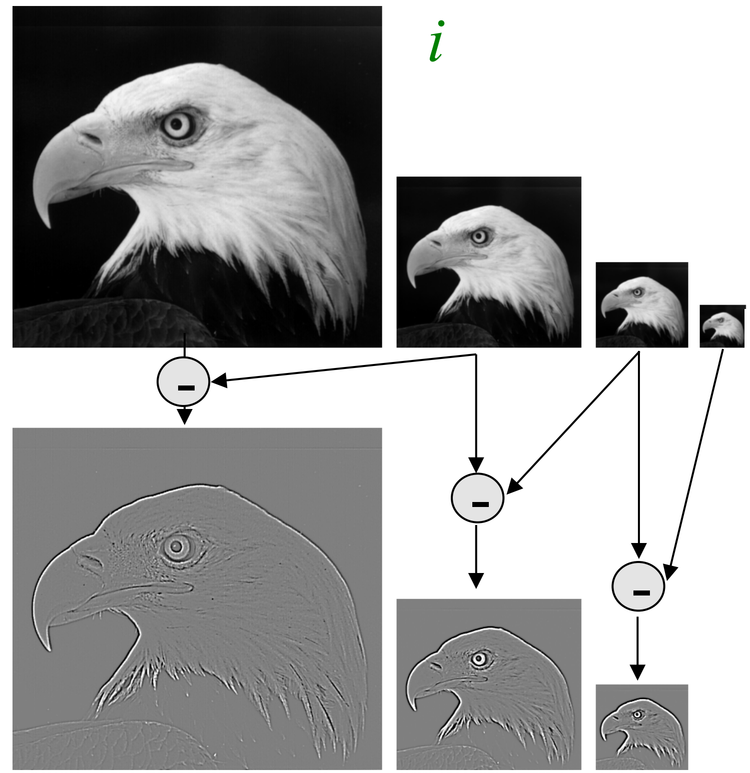
Intuition

- Scale space parses information according to spatial frequency content.
- By the sampling theorem, lower spatial frequencies can be captured with coarser spatial sampling.
- It should be possible to “subsample” the scale space components that correspond to lower frequencies.

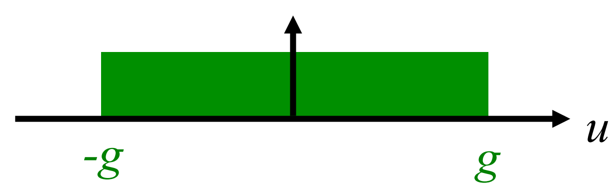
Formalization

- Lowpass Gaussian pyramids can be constructed by successively lowpass filtering the image and taking every other row and column (taking care that that the highest passed frequency can be properly captured).
- Bandpass Laplacian pyramids can be constructed by taking the pointwise difference of successive levels in the lowpass pyramid.
  - If Gaussian level i captures frequencies $|0\text{-}g|$
  - Gaussian level j captures frequencies $|0\text{-}f|$
  - Then the difference of i and j will cover frequencies $|g\text{-}f|$

Gaussian pyramid



Laplacian pyramid



Local scale: Pyramids

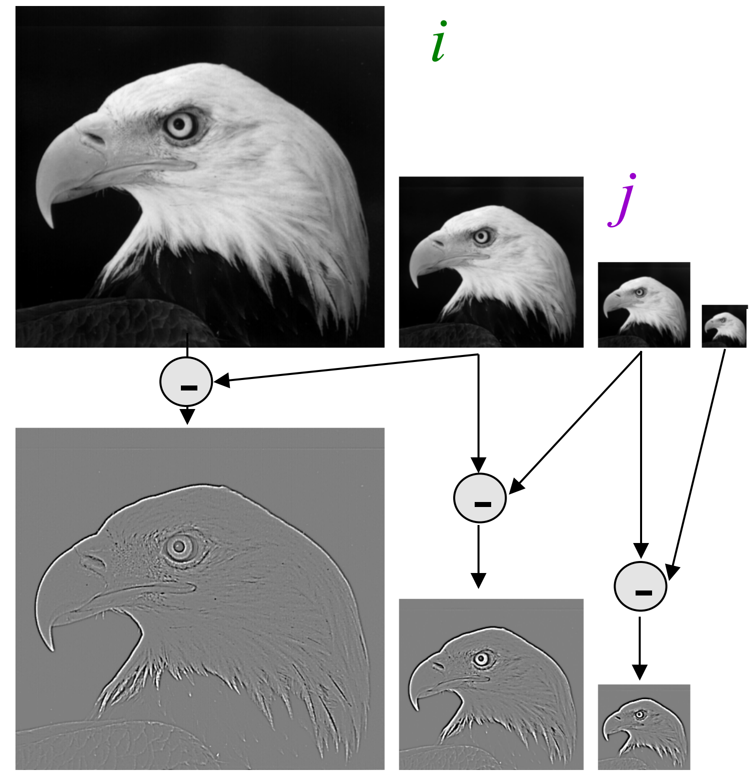
Intuition

- Scale space parses information according to spatial frequency content.
- By the sampling theorem, lower spatial frequencies can be captured with coarser spatial sampling.
- It should be possible to “subsample” the scale space components that correspond to lower frequencies.

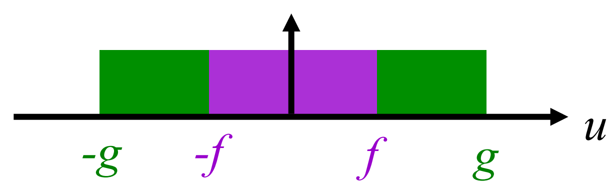
Formalization

- Lowpass Gaussian pyramids can be constructed by successively lowpass filtering the image and taking every other row and column (taking care that that the highest passed frequency can be properly captured).
- Bandpass Laplacian pyramids can be constructed by taking the pointwise difference of successive levels in the lowpass pyramid.
  - If Gaussian level i captures frequencies $|0\text{-}g|$
  - Gaussian level j captures frequencies $|0\text{-}f|$
  - Then the difference of i and j will cover frequencies $|g\text{-}f|$

Gaussian pyramid



Laplacian pyramid



Local scale: Pyramids

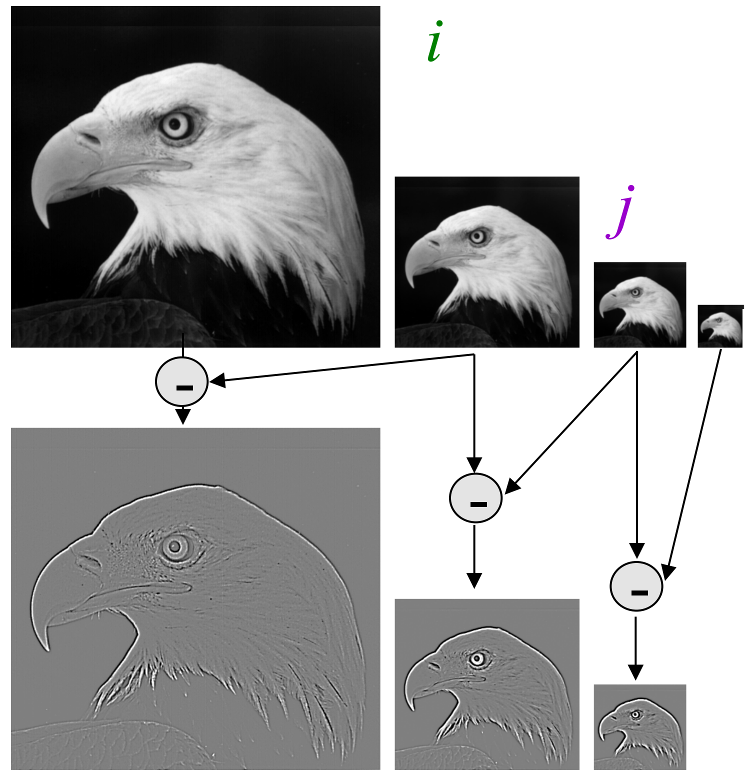
Intuition

- Scale space parses information according to spatial frequency content.
- By the sampling theorem, lower spatial frequencies can be captured with coarser spatial sampling.
- It should be possible to “subsample” the scale space components that correspond to lower frequencies.

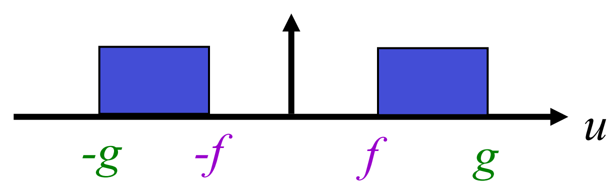
Formalization

- Lowpass Gaussian pyramids can be constructed by successively lowpass filtering the image and taking every other row and column (taking care that that the highest passed frequency can be properly captured).
- Bandpass Laplacian pyramids can be constructed by taking the pointwise difference of successive levels in the lowpass pyramid.
  - If Gaussian level i captures frequencies $|0\text{-}g|$
  - Gaussian level j captures frequencies $|0\text{-}f|$
  - Then the difference of i and j will cover frequencies $|g\text{-}f|$

Gaussian pyramid



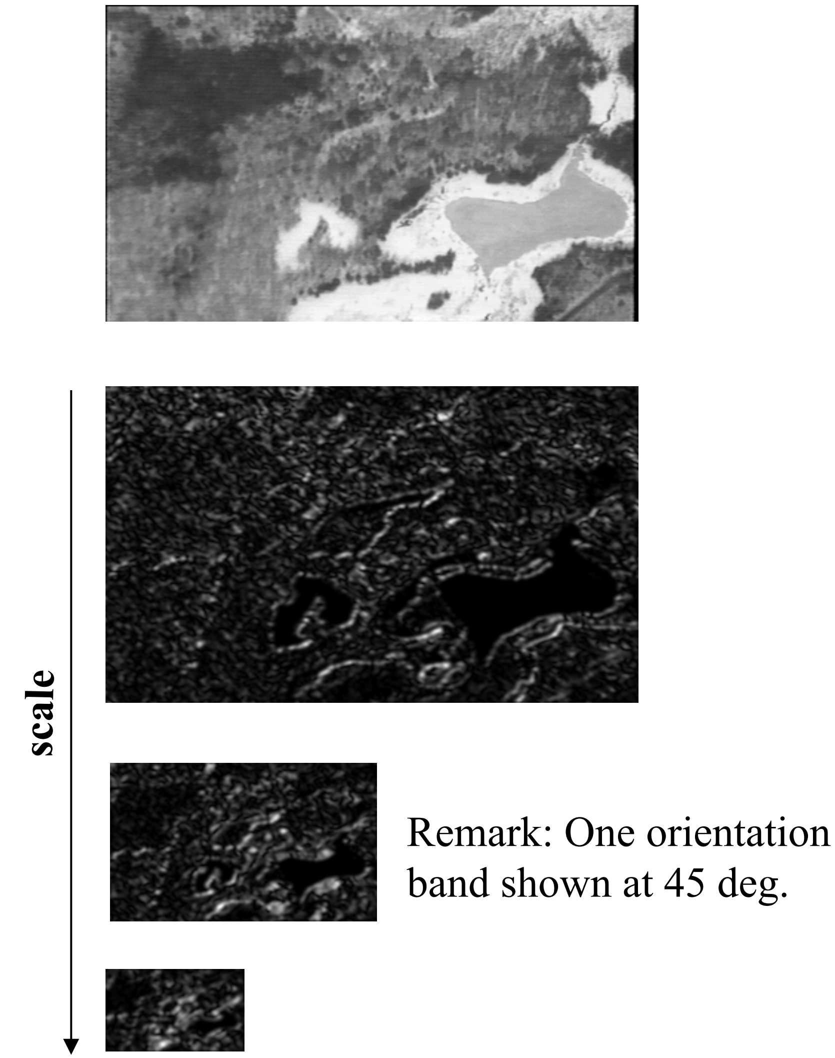
Laplacian pyramid



Local scale & orientation : Pyramids

Bringing it all together

- Application of oriented filters across pyramid levels allows us to build oriented pyramids.
- Now we have the ability to decompose an image according to local scale and orientation content…
- …in a storage efficient data structure.

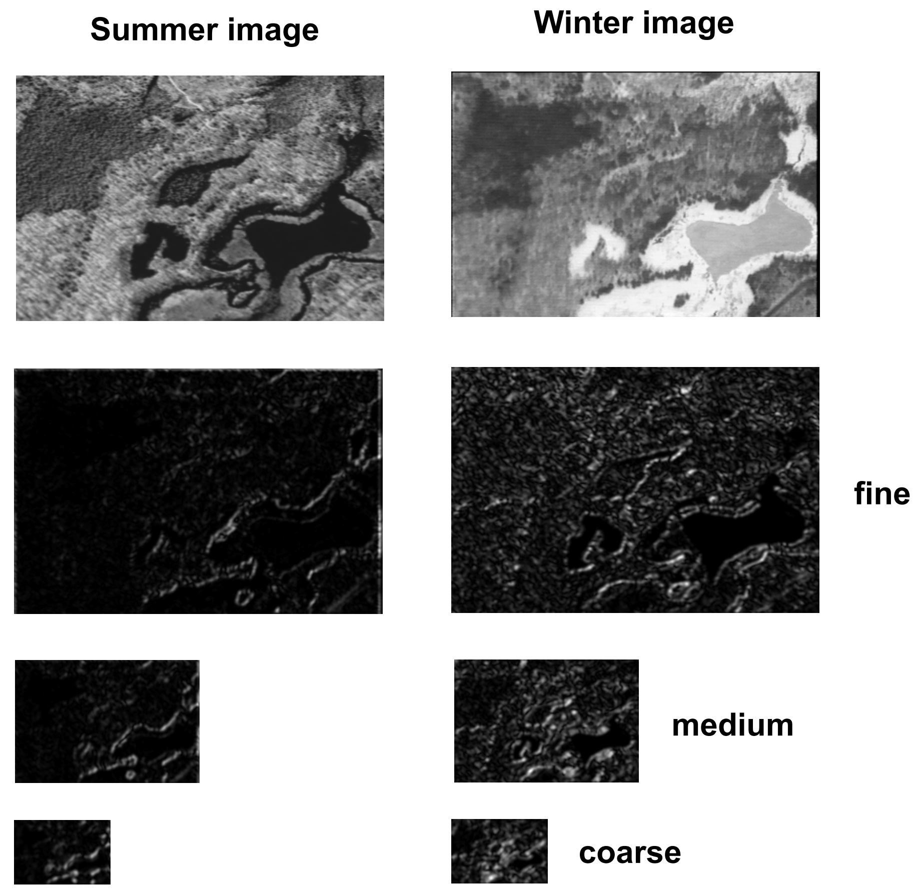


Oriented pyramid

Local scale & orientation: Toward invariance

Remark

- This type of representation can make local geometric similarity explicit…
- even in the presence of great photometric differences.
- For example: This type of representation can be an enabling component in matching images of the same scene across variable
  - Illumination
  - View direction
  - Surface cover



Remark: Only one of four orientations shown.

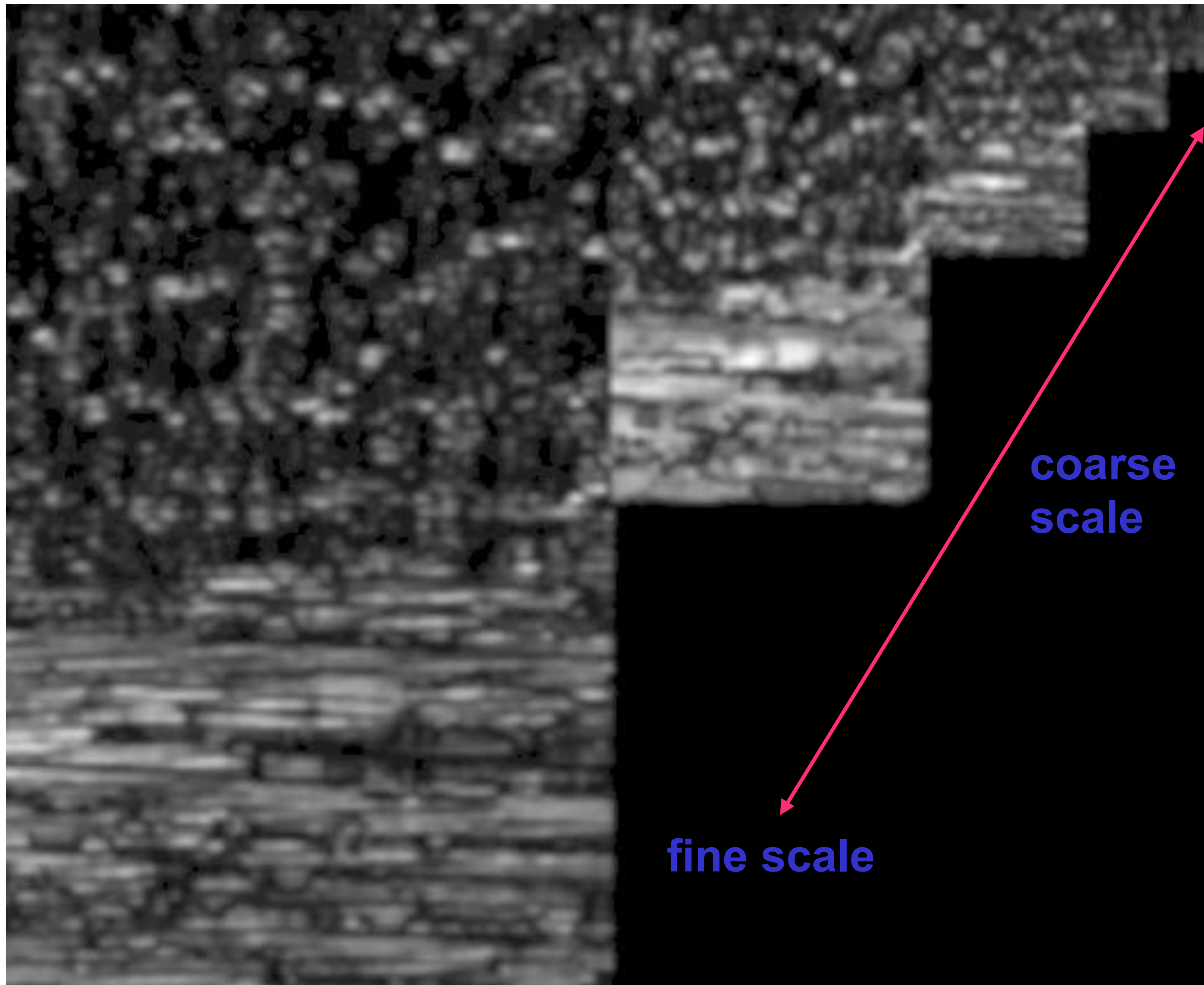
Local scale & orientation: Texture analysis

source

Local scale & orientation: Texture analysis

source

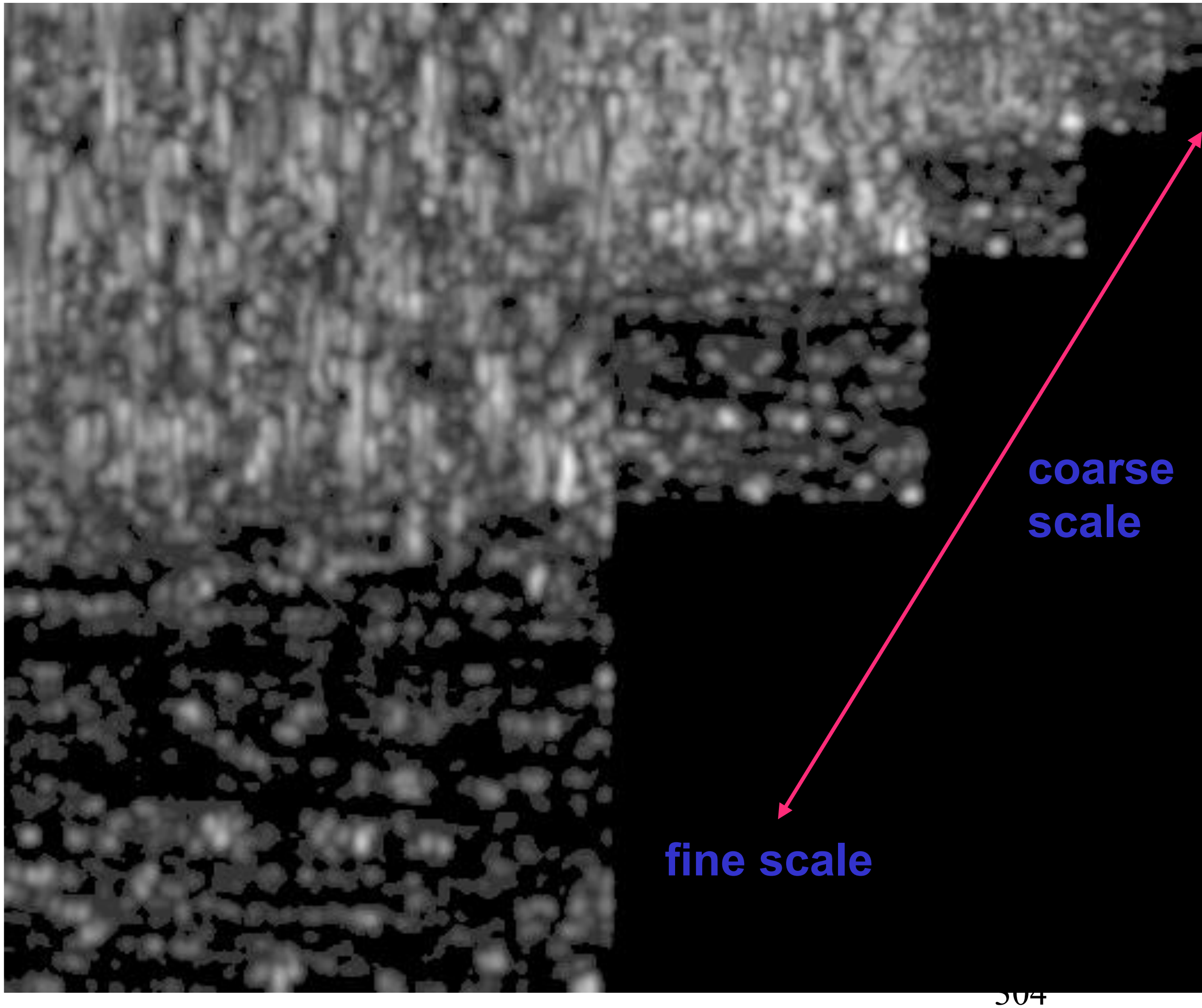
horizontal filtering result



Local scale & orientation: Texture analysis

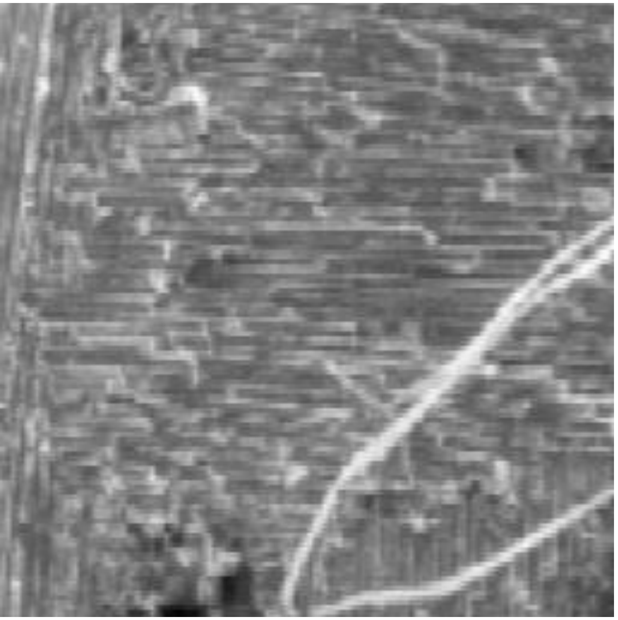
source

vertical filtering result



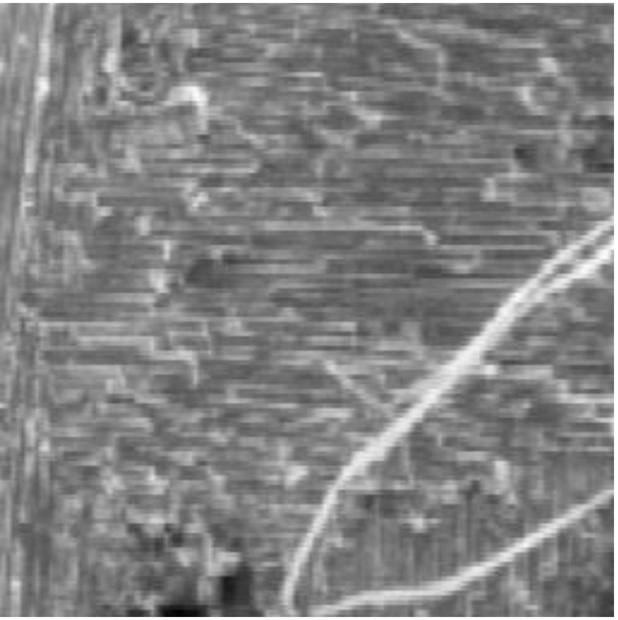
Local scale & orientation: Texture analysis

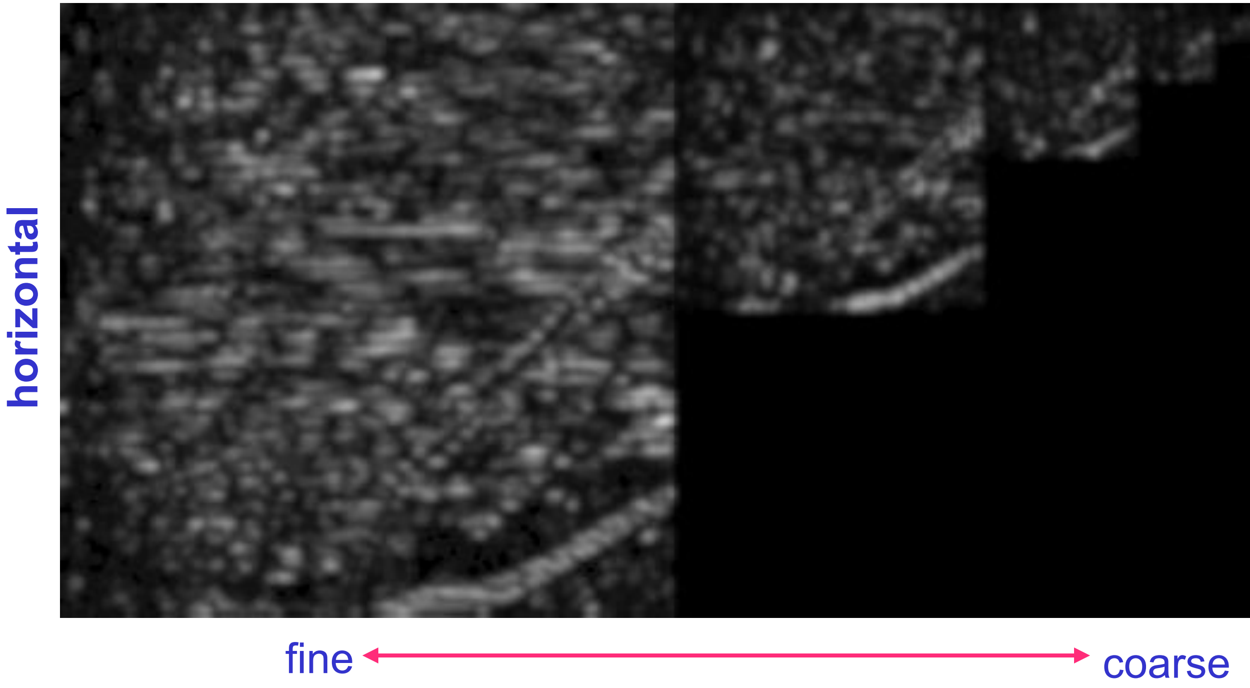
source



Local scale & orientation: Texture analysis

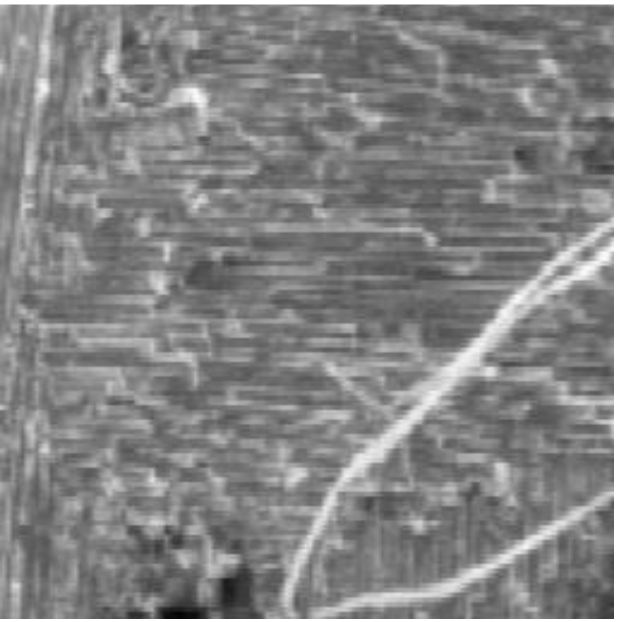
source



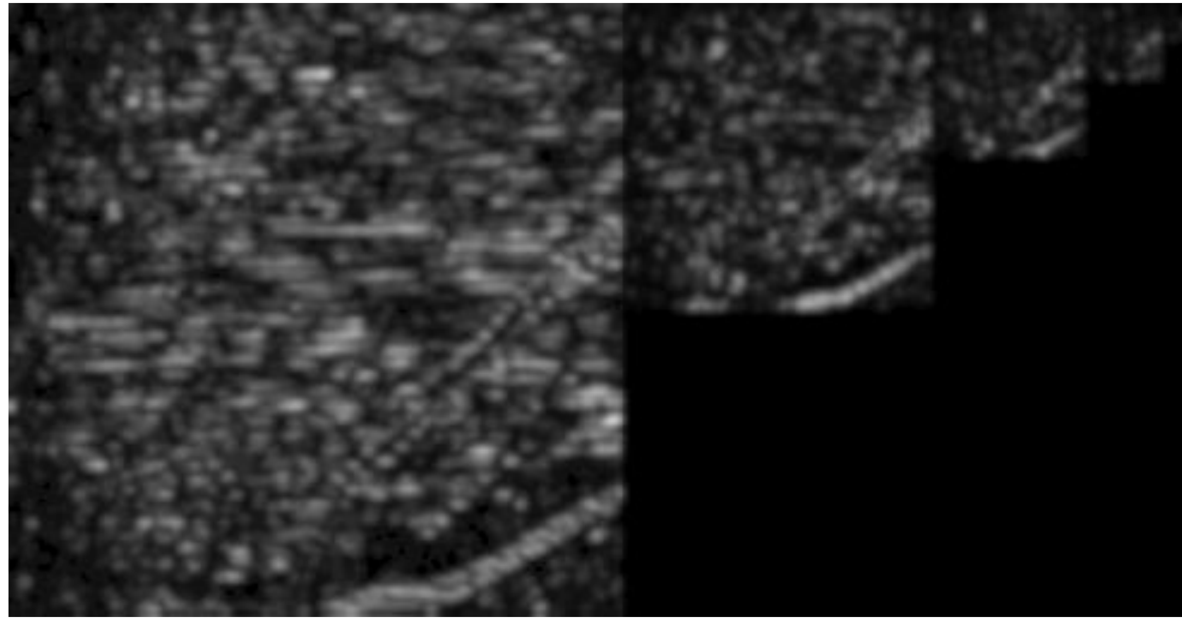


Local scale & orientation: Texture analysis

source

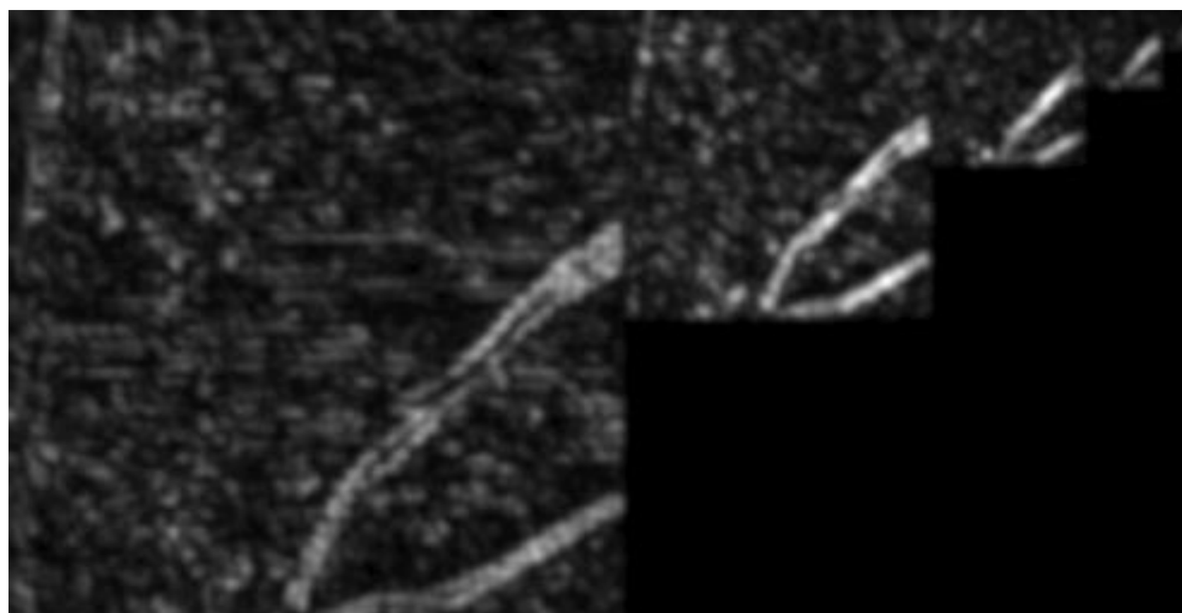


horizontal



fine ↔ coarse

diagonal



Summary

- Introduction
- Basics
- The Fourier transform
- Local operators
- Restoration and enhancement
- The discrete case
- Local scale and orientation