MATH/EECS 1028: Discrete Math for Engineers, Winter 2017

Tutorial 8 (Week of Mar 9, 2017)

Notes:

1. Assume $\mathbb{R}$ to denote the real numbers, $\mathbb{Z}$ to denote the set of integers $(\ldots, -2, -1, 0, 1, 2, \ldots)$ and $\mathbb{N}$ to denote the natural numbers $(1, 2, 3, \ldots)$.
2. Topics: Pigeonhole principle, Proofs, Induction.
3. Note to the TA: There will be no quiz this week.

Questions:

1. Let s be a positive integer. Prove that every interval $[s, 2s]$ contains a power of 2.
2. Prove using induction that for all $n \in \mathbb{N}$, 3 divides $4^n - 1$.
3. Prove using induction that for all $n \in \mathbb{N}$, the power set of an n-element set has 2^n elements.
4. Let $f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{x}{1-x}$. Define $f^2(x) = f(f(x)), f^3(x) = f(f(f(x)))$ and so on. Guess the form for $f^n(x)$ and prove your answer correct using induction on n.
5. Let k be a positive integer and x be real. Prove using induction that if $x + \frac{1}{x}$ is an integer then $x^k + \frac{1}{x^k}$ is also an integer.
6. Prove using mathematical induction that if n non-parallel straight lines on the plane intersect at a common point, they divide the plane into $2n$ regions.
7. Use strong induction to show that every positive integer can be written as a sum of distinct powers of 2, that is, as a subset of the integers $2^0 = 1, 2^1 = 2, 2^2 = 4$ and so on.
   Hint: For the inductive step, separately consider the case where $k+1$ is even and where it is odd. Where it is even, note that $\frac{k+1}{2}$ is an integer.

   Note: The question implies that every positive integer can be written in binary.