Math/CSE 1028: Discrete Mathematics for Engineers

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Course page: http://www.cse.yorku.ca/course/1028

Counting

Many applications. E.g.:

- How many factors does an integer have?
- How many trailing zeroes are there in 150! ?
- How many case-sensitive alphanumeric passwords are there of length k?
- How many binary functions with n binary inputs are there?

The product rule

- If 2 independent subtasks can be done in m, n ways (resp.) then the task can be done in mn ways.
- E.g. If I have 2 keyboard players and 3 percussionists, I can choose a keyboard-percussion duo in 6 ways.
- E.g. 2: How many 2 digit numbers are there?
- E.g. 3: No of k character alphanumeric passwords

Counting functions

Binary output:

- One binary input: 2^2 functions
- One integer input: 2^{MAXINT} functions
- n binary inputs: 2^{2^n} functions

Integer output:

- One integer input: $MAXINT^{MAXINT}$ functions

Binary strings

- Number of binary strings of length n?

  - easy

- Number of subsets of a set of n elements?

  Relate to previous question….

  Each subset is uniquely determined by a binary indicator string of length n….

Number of factors

- How many factors of 2^n are there?
- Wrong argument: each 2 may or may not be chosen….
- Correct argument: we can take 0,1,.., n of the 2's. Therefore n+1 factors (including 1 and 2^n itself).

Number of factors (general)

Q: How many factors of m are there?

A: Let $m = 2^a 3^b 5^c$….Then the number of factors (including 1 and m itself) is $(a+1)(b+1)(c+1)$....

Proof: we can take 0,1,.., a of the 2's, 0,1,.., b of the 3's and so on.

Powers of 2

Q: How many factors of 2 does 9! have?

A: 7

Q: How many factors of 2 does n! have?

A: (n div 2) + (n div 4) + (n div 8) +…

(div gives the integer quotient)

Number of trailing zeroes

Q: How many trailing zeroes in 150! ?

A: Equal to the number of factors of 10.

There are many more 2's than 5's so it is enough to count the number of 5's in the factorization. So the answer is

(150 div 5) + (150 div 25) + (150 div 125)

The sum rule

- If a job can be done in one of m ways or (exclusive or) in one of n ways, the total number of ways is m+n
- E.g. If you must take 3 credits of Math or 3 credits of Physics (but not both) an there are m Math courses and p Physics courses, there is a total of m+p courses to choose from.
- **Often used together with the product rule**

Counting strings

- What is the number of binary strings of length 4 containing exactly one 1?
- What is the number of 4 character DNA sequences containing exactly 1 A?

More complex problems

Q: How many 2 digit numbers are multiples of 11 or 13?

A: 9 (multiples of 11) + 7 (multiples of 13)

Harder question: How many 3 digit numbers are multiples of 11 or 13?

The problem is 143 (and its multiples) are multiples of both!

Inclusion-Exclusion (or the subtraction rule)

$|A \cup B| = |A|+|B| - |A \cap B|$

- e.g. How many 3 digit numbers are multiples of 11 or 13?
- A: No of 3 digit multiples of 11 + No of 3 digit multiples of 13 – No of 3 digit multiples of 143.
- In how many ways can you toss two dice, so that the first toss is a 1 OR the last toss is a 6?

A common trick

Q1: How many 5 element DNA sequences do not contain a C?

A: 3^5

Q2: How many 5 element DNA sequences contain at least one C?

Hint: Use the previous answer.

Q3: What is the number of length 5 alphanumeric strings with at least one digit?

Permutations

- Part of Combinatorics
- $P(n,r)$: number of ways in which r students (out of a class of n) can be lined up for a picture.
- $P(n,n) = n!$
- $P(n,r) = n!/(n-r)!$

Recall that $0! = 1$ by definition.

Combinations

- C(n,r): Number of ways r students can be chosen from a class on n students
- P(n,r) = C(n,r) P(r,r)
- C(n,r) = n!/[r!(n-r)!]

Examples

- Q22, pg 414: How many permutations of the letters ABCDEFG cointain the string BCD?
- How many binary strings of length n contain exactly k 1’s?
- Q 32, pg 414: How many strings of 6 lowercase letters contain the letter a?

Binomial Coefficients

$$(x+y)^n = \sum_{r=0}^{n} C(n,r)\, x^{n-r}\, y^r$$

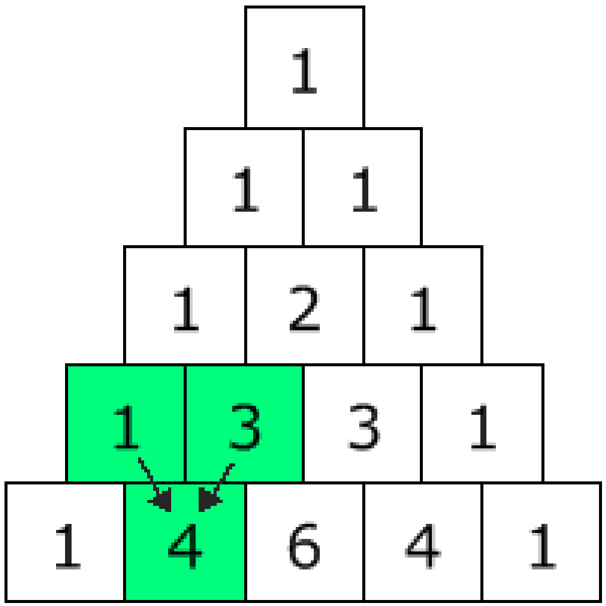
- We will not prove it formally
- It follows that $\sum C(n,r) = 2^n$
- And $\sum (-1)^n C(n,r) = 0$
- And $\sum C(n,r)\, 2^r = 3^n$

An important identity

- C(n,r)+C(n,r-1) = C(n+1,r)
- Direct proof (done on board)
- Combinatorial proof: Choosing r items from a set of n+1 items (details on board)
- Note: Use the above for computing C(n,r) in a program…it uses only additions and often avoids overflow issues

Pascal's triangle

- See http://www.mathsisfun.com/pascals-triangle.html for more facts



$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$