1. Prove whether each of the following is true or false.  \( x \) and \( y \) are real variables.

1) \( \forall x \exists y \ x \cdot y = 5 \)
2) \( \exists y \forall x \ x \cdot y = 5 \)
3) \( \forall x \exists y \ x \cdot y = 0 \)
4) \( \exists y \forall x \ x \cdot y = 0 \)
5) \( \exists a \forall x \exists y \ [x = a \text{ or } x \cdot y = 5] \)

2. Asymptotic Running Times

True or False? All logarithms are base 2. No justification is necessary.

(a) \( 5n^2 \log n \in O(n^2) \)
(b) \( 4^{8n} \in O(8^{4n}) \)
(c) \( 2^{10\log n + 100(\log n)^{11}} \in O(n^{10}) \)
(d) \( 2n^2 \log n + 3n^2 \in \Theta(n^3) \)

3. Big-Oh Definition

Fill in the blanks:
\( f(n) \in O(g(n)) \) iff \( c > 0, \ n_0 > 0 \), such that \( n > n_0 \), \( f(n) \leq cg(n) \)

4. Order the following functions by increasing asymptotic growth rate:

\[
\begin{align*}
4n \log n + 2n &< 2^{10} \log n \\
3n + 100 \log n &< 4n \\
n^2 + 10n &< n^3 \\
n \log n &< 2^n
\end{align*}
\]

5. Prove that \( n \log n - n = \Omega(n) \).

6. Prove that if \( d(n) = O(f(n)) \) and \( e(n) = O(g(n)) \), then the product \( d(n)e(n) = O(f(n)g(n)) \).

7. An evil king has \( n \) bottles of wine, and a spy has just poisoned one of them. Unfortunately, they dont know which one it is. The poison is very deadly; just one drop diluted even a billion to one will still kill. Even so, it takes a full month for the poison to take effect. Design a scheme for determining exactly which one of the wine bottles was poisoned in just one months time while expending only \( O(\log n) \) royal tasters. State your scheme briefly, in English.

8. Asymptotic Running Times

True or False? All logarithms are base 2. No justification is necessary.

(a) \( 2^n \in \Omega(n^3) \)
(b) \( 3n^3 + 17n^2 \in O(n^3) \)
(c) \( 5n^2 \log n \in O(n^2) \)
(d) \( 2^{10\log n + 100(\log n)^{11}} \in O(n^{10}) \)
(e) \( 2n^2 \log n + 3n^2 \in \Theta(n^3) \)

9. Show that \( n^2 \) is \( \Omega(n \log n) \).