

# Sequences and Summations

Section 2.4

# Section Summary

- Sequences.
  - Examples: Geometric Progression, Arithmetic Progression
- Recurrence Relations
  - Example: Fibonacci Sequence
- Summations
- Special Integer Sequences (*optional*)

# Introduction

- Sequences are ordered lists of elements.
  - 1, 2, 3, 5, 8
  - 1, 3, 9, 27, 81, .....
- Sequences arise throughout mathematics, computer science, and in many other disciplines, ranging from botany to music.
- We will introduce the terminology to represent sequences and sums of the terms in the sequences.

# Sequences

**Definition:** A *sequence* is a function from a subset of the integers (usually either the set  $\{0, 1, 2, 3, 4, \dots\}$  or  $\{1, 2, 3, 4, \dots\}$ ) to a set  $S$ .

- The notation  $a_n$  is used to denote the image of the integer  $n$ . We can think of  $a_n$  as the equivalent of  $f(n)$  where  $f$  is a function from  $\{0, 1, 2, \dots\}$  to  $S$ . We call  $a_n$  a *term* of the sequence.

# Sequences

**Example:** Consider the sequence  $\{a_n\}$  where

$$a_n = \frac{1}{n} \quad \{a_n\} = \{a_1, a_2, a_3, \dots\}$$

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots$$

# Geometric Progression

**Definition:** A *geometric progression* is a sequence of the form:  $a, ar, ar^2, \dots, ar^n, \dots$

where the *initial term*  $a$  and the *common ratio*  $r$  are real numbers.

## Examples:

1. Let  $a = 1$  and  $r = -1$ . Then:

$$\{b_n\} = \{b_0, b_1, b_2, b_3, b_4, \dots\} = \{1, -1, 1, -1, 1, \dots\}$$

2. Let  $a = 2$  and  $r = 5$ . Then:

$$\{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \dots\} = \{2, 10, 50, 250, 1250, \dots\}$$

3. Let  $a = 6$  and  $r = 1/3$ . Then:

$$\{d_n\} = \{d_0, d_1, d_2, d_3, d_4, \dots\} = \{6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \dots\}$$

# Arithmetic Progression

**Definition:** A *arithmetic progression* is a sequence of the form:  $a, a + d, a + 2d, \dots, a + nd, \dots$

where the *initial term*  $a$  and the *common difference*  $d$  are real numbers.

**Examples:**

1. Let  $a = -1$  and  $d = 4$ :

$$\{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \dots\} = \{-1, 3, 7, 11, 15, \dots\}$$

2. Let  $a = 7$  and  $d = -3$ :

$$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \dots\} = \{7, 4, 1, -2, -5, \dots\}$$

3. Let  $a = 1$  and  $d = 2$ :

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \dots\} = \{1, 3, 5, 7, 9, \dots\}$$

# Strings

**Definition:** A *string* is a finite sequence of characters from a finite set (an alphabet).

- Sequences of characters or bits are important in computer science.
- The *empty string* is represented by  $\lambda$ .
- The string *abcde* has *length* 5.

# Recurrence Relations

**Definition:** A *recurrence relation* for the sequence  $\{a_n\}$  is an equation that expresses  $a_n$  in terms of one or more of the previous terms of the sequence, namely,  $a_0, a_1, \dots, a_{n-1}$ , for all integers  $n$  with  $n \geq n_0$ , where  $n_0$  is a nonnegative integer.

- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
- The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

# Questions about Recurrence Relations

**Example 1:** Let  $\{a_n\}$  be a sequence that satisfies the recurrence relation  $a_n = a_{n-1} + 3$  for  $n = 1, 2, 3, 4, \dots$  and suppose that  $a_0 = 2$ . What are  $a_1$ ,  $a_2$  and  $a_3$ ?

[Here  $a_0 = 2$  is the initial condition.]

**Solution:** We see from the recurrence relation that

$$a_1 = a_0 + 3 = 2 + 3 = 5$$

$$a_2 = 5 + 3 = 8$$

$$a_3 = 8 + 3 = 11$$

## Questions about Recurrence Relations

**Example 2:** Let  $\{a_n\}$  be a sequence that satisfies the recurrence relation  $a_n = a_{n-1} - a_{n-2}$  for  $n = 2, 3, 4, \dots$  and suppose that  $a_0 = 3$  and  $a_1 = 5$ . What are  $a_2$  and  $a_3$ ?  
[Here the initial conditions are  $a_0 = 3$  and  $a_1 = 5$ .]

**Solution:** We see from the recurrence relation that

$$a_2 = a_1 - a_0 = 5 - 3 = 2$$

$$a_3 = a_2 - a_1 = 2 - 5 = -3$$

# Fibonacci Sequence

**Definition:** Define the *Fibonacci sequence*,  $f_0, f_1, f_2, \dots$ , by:

- Initial Conditions:  $f_0 = 0, f_1 = 1$
- Recurrence Relation:  $f_n = f_{n-1} + f_{n-2}$

**Example:** Find  $f_2, f_3, f_4, f_5$  and  $f_6$  .

**Answer:**

$$f_2 = f_1 + f_0 = 1 + 0 = 1,$$

$$f_3 = f_2 + f_1 = 1 + 1 = 2,$$

$$f_4 = f_3 + f_2 = 2 + 1 = 3,$$

$$f_5 = f_4 + f_3 = 3 + 2 = 5,$$

$$f_6 = f_5 + f_4 = 5 + 3 = 8.$$



# Solving Recurrence Relations

- Finding a formula for the  $n$ th term of the sequence generated by a recurrence relation is called *solving the recurrence relation*.
- Such a formula is called a *closed formula*.
- Various methods for solving recurrence relations will be covered in Chapter 8 where recurrence relations will be studied in greater depth.
- Here we illustrate by example the method of iteration in which we need to guess the formula. The guess can be proved correct by the method of induction (Chapter 5).

# Iterative Solution Example

**Method 1:** Working upward, forward substitution

Let  $\{a_n\}$  be a sequence that satisfies the recurrence relation  $a_n = a_{n-1} + 3$  for  $n = 2, 3, 4, \dots$  and suppose that  $a_1 = 2$ .

$$a_2 = 2 + 3$$

$$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$

$$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$$

.

.

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$$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n - 2)) + 3 = 2 + 3(n - 1)$$

# Iterative Solution Example

**Method 2:** Working downward, backward substitution

Let  $\{a_n\}$  be a sequence that satisfies the recurrence relation  $a_n = a_{n-1} + 3$  for  $n = 2, 3, 4, \dots$  and suppose that  $a_1 = 2$ .

$$\begin{aligned} a_n &= a_{n-1} + 3 \\ &= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\ &= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \cdot \\ &= a_2 + 3(n-2) = (a_1 + 3) + 3(n-2) = 2 + 3(n-1) \end{aligned}$$

# Financial Application

**Example:** Suppose that a person deposits \$10,000.00 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Let  $P_n$  denote the amount in the account after  $n$  years.  $P_n$  satisfies the following recurrence relation:

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11) P_{n-1}$$

with the initial condition  $P_0 = 10,000$

*Continued on next slide →*

# Financial Application

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11) P_{n-1}$$

with the initial condition  $P_0 = 10,000$

**Solution:** Forward Substitution

$$P_1 = (1.11)P_0$$

$$P_2 = (1.11)P_1 = (1.11)^2P_0$$

$$P_3 = (1.11)P_2 = (1.11)^3P_0$$

⋮

$$P_n = (1.11)P_{n-1} = (1.11)^nP_0 = (1.11)^n 10,000$$

$$P_n = (1.11)^n 10,000 \text{ (Can prove by induction, covered in Chapter 5)}$$

$$P_{30} = (1.11)^{30} 10,000 = \$228,992.97$$



# Special Integer Sequences (*opt*)

- Given a few terms of a sequence, try to identify the sequence. Conjecture a formula, recurrence relation, or some other rule.
- Some questions to ask?
  - Are there repeated terms of the same value?
  - Can you obtain a term from the previous term by adding an amount or multiplying by an amount?
  - Can you obtain a term by combining the previous terms in some way?
  - Are they cycles among the terms?
  - Do the terms match those of a well known sequence?

# Questions on Special Integer Sequences (*opt*)

**Example 1:** Find formulae for the sequences with the following first five terms:  $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$

**Solution:** Note that the denominators are powers of 2. The sequence with  $a_n = 1/2^n$  is a possible match. This is a geometric progression with  $a = 1$  and  $r = \frac{1}{2}$ .

**Example 2:** Consider  $1, 3, 5, 7, 9$

**Solution:** Note that each term is obtained by adding 2 to the previous term. A possible formula is  $a_n = 2n + 1$ . This is an arithmetic progression with  $a = 1$  and  $d = 2$ .

**Example 3:**  $1, -1, 1, -1, 1$

**Solution:** The terms alternate between 1 and -1. A possible sequence is  $a_n = (-1)^n$ . This is a geometric progression with  $a = 1$  and  $r = -1$ .

# Useful Sequences

**TABLE 1** Some Useful Sequences.

<i>n</i> th Term	First 10 Terms
$n^2$	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
$n^3$	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
$n^4$	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
$2^n$	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
$3^n$	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...
$f_n$	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

# Guessing Sequences (*optional*)

**Example:** Conjecture a simple formula for  $a_n$  if the first 10 terms of the sequence  $\{a_n\}$  are 1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047.

**Solution:** Note the ratio of each term to the previous approximates 3. So now compare with the sequence  $3^n$ . We notice that the  $n$ th term is 2 less than the corresponding power of 3. So a good conjecture is that  $a_n = 3^n - 2$ .

# Integer Sequences (*optional*)

- Integer sequences appear in a wide range of contexts. Later we will see the sequence of prime numbers (Chapter 4), the number of ways to order  $n$  discrete objects (Chapter 6), the number of moves needed to solve the Tower of Hanoi puzzle with  $n$  disks (Chapter 8), and the number of rabbits on an island after  $n$  months (Chapter 8).
- Integer sequences are useful in many fields such as biology, engineering, chemistry and physics.
- On-Line Encyclopedia of Integer Sequences (OESIS) contains over 200,000 sequences. Began by Neil Stone in the 1960s (printed form). Now found at <http://oeis.org/Spuzzle.html>

# Integer Sequences (*optional*)

- Here are three interesting sequences to try from the OESIS site. To solve each puzzle, find a rule that determines the terms of the sequence.
- Guess the rules for forming for the following sequences:
  - 2, 3, 3, 5, 10, 13, 39, 43, 172, 177, ...
    - Hint: Think of adding and multiplying by numbers to generate this sequence.
  - 0, 0, 0, 0, 4, 9, 5, 1, 1, 0, 55, ...
    - Hint: Think of the English names for the numbers representing the position in the sequence and the Roman Numerals for the same number.
  - 2, 4, 6, 30, 32, 34, 36, 40, 42, 44, 46, ...
    - Hint: Think of the English names for numbers, and whether or not they have the letter 'e.'
- The answers and many more can be found at <http://oeis.org/Spuzzle.html>

# Summations

- Sum of the terms  $a_m, a_{m+1}, \dots, a_n$  from the sequence  $\{a_n\}$
- The notation:

$$\sum_{j=m}^n a_j \quad \sum_{j=m}^n a_j \quad \sum_{m \leq j \leq n} a_j$$

represents

$$a_m + a_{m+1} + \dots + a_n$$

- The variable  $j$  is called the *index of summation*. It runs through all the integers starting with its *lower limit*  $m$  and ending with its *upper limit*  $n$ .

# Summations

- More generally for a set  $S$ :

$$\sum_{j \in S} a_j$$

- **Examples:**

$$r^0 + r^1 + r^2 + r^3 + \dots + r^n = \sum_0^n r^j$$

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots = \sum_1^{\infty} \frac{1}{i}$$

If  $S = \{2, 5, 7, 10\}$  then  $\sum_{j \in S} a_j = a_2 + a_5 + a_7 + a_{10}$

# Product Notation (*optional*)

- Product of the terms  $a_m, a_{m+1}, \dots, a_n$   
from the sequence  $\{a_n\}$

- The notation:

$$\prod_{j=m}^n a_j \quad \prod_{j=m}^n a_j \quad \prod_{m \leq j \leq n} a_j$$

represents

$$a_m \times a_{m+1} \times \dots \times a_n$$

# Geometric Series

Sums of terms of geometric progressions

$$\sum_{j=0}^n ar^j = \begin{cases} \frac{ar^{n+1} - a}{r-1} & r \neq 1 \\ (n+1)a & r = 1 \end{cases}$$

**Proof:** Let  $S_n = \sum_{j=0}^n ar^j$  To compute  $S_n$ , first multiply both sides of the equality by  $r$  and then manipulate the resulting sum as follows:

$$\begin{aligned} rS_n &= r \sum_{j=0}^n ar^j \\ &= \sum_{j=0}^n ar^{j+1} \end{aligned}$$

*Continued on next slide →*

# Geometric Series

$$= \sum_{j=0}^n ar^{j+1} \quad \text{From previous slide.}$$

$$= \sum_{k=1}^{n+1} ar^k \quad \text{Shifting the index of summation with } k = j + 1.$$

$$= \left( \sum_{k=0}^n ar^k \right) + (ar^{n+1} - a) \quad \text{Removing } k = n + 1 \text{ term and adding } k = 0 \text{ term.}$$

$$= S_n + (ar^{n+1} - a) \quad \text{Substituting } S \text{ for summation formula}$$

∴

$$rS_n = S_n + (ar^{n+1} - a)$$

$$S_n = \frac{ar^{n+1} - a}{r - 1} \quad \text{if } r \neq 1$$

$$S_n = \sum_{j=0}^n ar^j = \sum_{j=0}^n a = (n + 1)a \quad \text{if } r = 1$$

# Some Useful Summation Formulae

**TABLE 2** Some Useful Summation Formulae.

<i>Sum</i>	<i>Closed Form</i>
$\sum_{k=0}^n ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^n k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^n k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^n k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k,  x  < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1},  x  < 1$	$\frac{1}{(1-x)^2}$

Geometric Series: We just proved this.

Later we will prove some of these by induction.

Proof in text (requires calculus)

# Cardinality of Sets

Section 2.5



# Section Summary

- Cardinality
- Countable Sets
- Computability

# Cardinality

**Definition:** The *cardinality* of a set  $A$  is equal to the cardinality of a set  $B$ , denoted

$$|A| = |B|,$$

if and only if there is a one-to-one correspondence (*i.e.*, a bijection) from  $A$  to  $B$ .

- If there is a one-to-one function (*i.e.*, an injection) from  $A$  to  $B$ , the cardinality of  $A$  is less than or the same as the cardinality of  $B$  and we write  $|A| \leq |B|$ .
- When  $|A| \leq |B|$  and  $A$  and  $B$  have different cardinality, we say that the cardinality of  $A$  is less than the cardinality of  $B$  and write  $|A| < |B|$ .

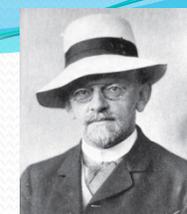
# Cardinality

- **Definition:** A set that is either finite or has the same cardinality as the set of positive integers ( $\mathbf{Z}^+$ ) is called *countable*. A set that is not countable is *uncountable*.
- The set of real numbers  $\mathbf{R}$  is an uncountable set.
- When an infinite set is countable (*countably infinite*) its cardinality is  $\aleph_0$  (where  $\aleph$  is aleph, the 1<sup>st</sup> letter of the Hebrew alphabet). We write  $|S| = \aleph_0$  and say that  $S$  has cardinality “aleph null.”

# Showing that a Set is Countable

- An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by the positive integers).
- The reason for this is that a one-to-one correspondence  $f$  from the set of positive integers to a set  $S$  can be expressed in terms of a sequence  $a_1, a_2, \dots, a_n, \dots$  where  $a_1 = f(1), a_2 = f(2), \dots, a_n = f(n), \dots$

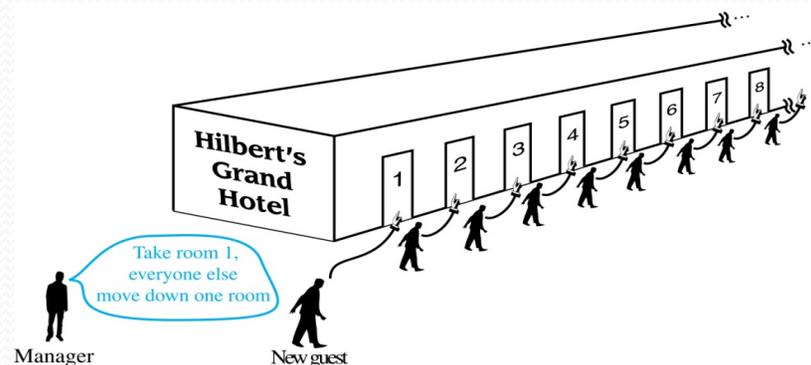
# Hilbert's Grand Hotel



David Hilbert

The Grand Hotel (example due to David Hilbert) has countably infinite number of rooms, each occupied by a guest. We can always accommodate a new guest at this hotel. How is this possible?

**Explanation:** Because the rooms of Grand Hotel are countable, we can list them as Room 1, Room 2, Room 3, and so on. When a new guest arrives, we move the guest in Room 1 to Room 2, the guest in Room 2 to Room 3, and in general the guest in Room  $n$  to Room  $n + 1$ , for all positive integers  $n$ . This frees up Room 1, which we assign to the new guest, and all the current guests still have rooms.

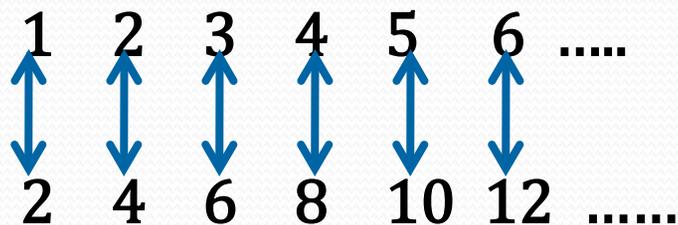


The hotel can also accommodate a countable number of new guests, all the guests on a countable number of buses where each bus contains a countable number of guests (see exercises).

# Showing that a Set is Countable

**Example 1:** Show that the set of positive even integers  $E$  is countable set.

**Solution:** Let  $f(x) = 2x$ .



Then  $f$  is a bijection from  $\mathbf{N}$  to  $E$  since  $f$  is both one-to-one and onto. To show that it is one-to-one, suppose that  $f(n) = f(m)$ . Then  $2n = 2m$ , and so  $n = m$ . To see that it is onto, suppose that  $t$  is an even positive integer. Then  $t = 2k$  for some positive integer  $k$  and  $f(k) = t$ . ◀

# Showing that a Set is Countable

**Example 2:** Show that the set of integers  $\mathbf{Z}$  is countable.

**Solution:** Can list in a sequence:

0, 1, -1, 2, -2, 3, -3, .....

Or can define a bijection from  $\mathbf{N}$  to  $\mathbf{Z}$ :

- When  $n$  is even:  $f(n) = n/2$
- When  $n$  is odd:  $f(n) = -(n-1)/2$



# The Positive Rational Numbers are Countable

- **Definition:** A *rational number* can be expressed as the ratio of two integers  $p$  and  $q$  such that  $q \neq 0$ .
  - $\frac{3}{4}$  is a rational number
  - $\sqrt{2}$  is not a rational number.

**Example 3:** Show that the positive rational numbers are countable.

**Solution:** The positive rational numbers are countable since they can be arranged in a sequence:

$$r_1, r_2, r_3, \dots$$

The next slide shows how this is done.

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# Strings

**Example 4:** Show that the set of finite strings  $S$  over a finite alphabet  $A$  is countably infinite.

Assume an alphabetical ordering of symbols in  $A$

**Solution:** Show that the strings can be listed in a sequence. First list

1. All the strings of length 0 in alphabetical order.
2. Then all the strings of length 1 in lexicographic (as in a dictionary) order.
3. Then all the strings of length 2 in lexicographic order.
4. And so on.

This implies a bijection from  $\mathbf{N}$  to  $S$  and hence it is a countably infinite set.



# The set of all Java programs is countable.

**Example 5:** Show that the set of all Java programs is countable.

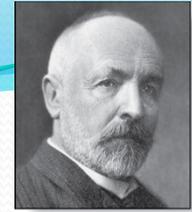
**Solution:** Let  $S$  be the set of strings constructed from the characters which can appear in a Java program. Use the ordering from the previous example. Take each string in turn:

- Feed the string into a Java compiler. (A Java compiler will determine if the input program is a syntactically correct Java program.)
- If the compiler says YES, this is a syntactically correct Java program, we add the program to the list.
- We move on to the next string.

In this way we construct an implied bijection from  $\mathbb{N}$  to the set of Java programs. Hence, the set of Java programs is countable.

# The Real Numbers are Uncountable

Georg Cantor  
(1845-1918)



**Example:** Show that the set of real numbers is uncountable.

**Solution:** The method is called the Cantor diagonalization argument, and is a proof by contradiction.

1. Suppose  $\mathbf{R}$  is countable. Then the real numbers between 0 and 1 are also countable (any subset of a countable set is countable - an exercise in the text).
2. The real numbers between 0 and 1 can be listed in order  $r_1, r_2, r_3, \dots$ .
3. Let the decimal representation of this listing be
$$\begin{aligned} r_1 &= 0.d_{11}d_{12}d_{13}d_{14}d_{15}d_{16}\dots \\ r_2 &= 0.d_{21}d_{22}d_{23}d_{24}d_{25}d_{26}\dots \\ r_3 &= 0.d_{31}d_{32}d_{33}d_{34}d_{35}d_{36}\dots \\ &\vdots \end{aligned}$$
4. Form a new real number with the decimal expansion  $r = .r_1r_2r_3r_4\dots$   
where  $r_i = 3$  if  $d_{ii} \neq 3$  and  $r_i = 4$  if  $d_{ii} = 3$
5.  $r$  is not equal to any of the  $r_1, r_2, r_3, \dots$ . Because it differs from  $r_i$  in its  $i$ th position after the decimal point. Therefore there is a real number between 0 and 1 that is not on the list since every real number has a unique decimal expansion. Hence, all the real numbers between 0 and 1 cannot be listed, so the set of real numbers between 0 and 1 is uncountable.
6. Since a set with an uncountable subset is uncountable (an exercise), the set of real numbers is uncountable.





# Computability (Optional)

- **Definition:** We say that a function is **computable** if there is a computer program in some programming language that finds the values of this function. If a function is not computable we say it is **uncomputable**.
- There are uncomputable functions. We have shown that the set of Java programs is countable. Exercise 38 in the text shows that there are uncountably many different functions from a particular countably infinite set (i.e., the positive integers) to itself. Therefore (Exercise 39) there must be uncomputable functions.