EECS-1019c: Assignment #10

Out of N points.

Section 8.1 [6pt]

11. [6pt]

a. [2pt] Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time.

> $a_n = a_{n-1} + a_{n-2}$ *for* $n \geq 2$.

b. [2pt] What are the initial conditions?

> $a_0 = 1$, $a_1 = 1$.

c. [2pt] In how many ways can this person climb a flight of eight stairs?

> $a_8 = f_9 = 34$.

Section 8.2 [24pt]

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4. [10pt] Solve these recurrence relations together with the initial conditions given.

a. [2pt] $a_n = a_{n-1} + 6a_{n-2}$ for $n \geq 2$, $a_0 = 3$, $a_1 = 6$.

> *Characteristic equation:* $r^2 - r - 6$, *so roots are* -5 *and* 1. $\alpha_1 + \alpha_2 = a_0 = 3$, $-5\alpha_1 + \alpha_2 = a_1 = 6$. *Solving,* $\alpha_1 = -\frac{1}{2}$, $\alpha_2 = 3\frac{1}{2}$.
> $\therefore a_n = -\frac{1}{2}(-5)^n + 3\frac{1}{2}$.

b. [2pt] $a_n = 7a_{n-1} - 10a_{n-2}$ for $n \geq 2$, $a_0 = 2$, $a_1 = 1$.

> *Characteristic equation:* $r^2 - 7r + 10$, *so roots are* -5 *and* -2.
> $\alpha_1 + \alpha_2 = a_0 = 2$, $-5\alpha_1 - 2\alpha_2 = a_1 = 1$. *Solving,* $\alpha_1 = -\frac{5}{3}$, $\alpha_2 = \frac{11}{3}$.
> $\therefore a_n = -\frac{5}{3}(-5)^n + \frac{11}{3}(-2)^n$.

c. [2pt] $a_n = 6a_{n-1} - 8a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 10$.

> *Characteristic equation:* $r^2 - 6r + 8$, *so roots are* -4 *and* -2.
> $\alpha_1 + \alpha_2 = a_0 = 4$, $-4\alpha_1 - 2\alpha_2 = a_1 = 10$. *Solving,* $\alpha_1 = -9$, $\alpha_2 = 13$.
> $\therefore a_n = -9(-4)^n + 13(-2)^n$.

d. [2pt] $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 2$, $a_0 = 4$, $a_1 = 1$.

> *Characteristic equation:* $r^2 - 2r + 1$, *so just a single root of* -1 *with a multiple of* 2.
> $\alpha_1 + 0\alpha_2 = a_0 = 4$, $-\alpha_1 - 1\alpha_2 = a_1 = 1$. *Solving,* $\alpha_1 = 4$, $\alpha_2 = 5$.
> $\therefore a_n = 4(-1)^n - 5n(-1)^n$.

e. [2pt] $a_n = a_{n-2}$ for $n \geq 2$, $a_0 = 5$, $a_1 = -1$.

> *Characteristic equation:* $r^2 - 1$, *so roots are* -1 *and* 1.
> $\alpha_1 + \alpha_2 = a_0 = 5$, $-\alpha_1 + \alpha_2 = a_1 = -1$. *Solving,* $\alpha_1 = 3$, $\alpha_2 = 2$.
> $\therefore a_n = 3(-1)^n + 2$.

8. [4pt] A model for the number of lobsters caught per year is based on the assumption that the number of lobsters caught in a year is the average of the number caught in the two previous years.

a. [2pt] Find a recurrence relation for $\{L_n\}$, where L_n is the number of lobsters caught in year n, under the assumption for this model.

> $L_n = \frac{1}{2}L_{n-1} + \frac{1}{2}L_{n-2}$, *for* $n > 2$.

b. [2pt] Find L_n if 100,000 lobsters were caught in year 1 and 300,000 were caught in year 2.

> *Characteristic equation is* $r^2 - \frac{1}{2}r - \frac{1}{2} = 0$, *or* $2r^2 - r - 1 = 0$. *The roots are* $1, -\frac{1}{2}$. *Thus, we know* $\alpha_1 - \frac{1}{2}\alpha_2 = 100000$ *and* $\alpha_1 + \frac{1}{4}\alpha_2 = 300000$ *Solving, we get* $\alpha_1 = 700000/3$ *and* $\alpha_2 = 800000/3$.
> $\therefore L_n = 233333.33 + 266666.67(-frac12)^n$

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12. [4pt] Find the solution to $a_n = 2a_{n-1} + a_{n-2}2a_{n-3}$. for $n = 3, 4, 5, \ldots$, with $a_0 = 3$, $a_1 = 6$, and $a_2 = 0$.

> *The characteristic equation is* $r^3 - 2r^2 - r + 2 = 0$. *The roots are* -1, 1, *and* 2. *Thus,* $\alpha_1 + \alpha_2 + \alpha_3 = a_0 = 3$, $-\alpha_1 + \alpha_2 + 2\alpha_3 = a_1 = 6$, *and* $\alpha_1 + \alpha_2 + 4\alpha_3 = a_2 = 0$. *Solving,* $\alpha_1 = -2$, $\alpha_2 = 6$, *and* $\alpha_3 = -1$.
> $\therefore a_n = -2(-1)^n + 6 - (2)^n$.

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24. [6pt] Consider the nonhomogeneous linear recurrence relation $a_n = 2a_{n1} + 2^n$.

a. [2pt] Show that $a_n = 2^{n+1}$ is a solution of this recurrence relation.

> *Solving the corresponding homogeneous, the characteristic equation is* $r - 2 = 0$, *with a root of* 2. *Thus,* $a_n^{(h)} = \alpha 2^n$.
> *Because* $f(n) = 2^n$, *since* 2^n *appears in our homogeneous solution with a multiplicity of* 1, *a reasonable trial solution is* $a_n^{(p)} = cn2^n$. *Thus,* $cn2^n = 2c(n-1)2^{n-1} + 2^n = c(n-1)2^n + 2^n = cn2^n - c2^n + 2^n$, *so* $c = 1$. *Hence, a particular equation is* $a_n^{(p)} = n2^n$.
> *So, the full solution is* $a_n = a_n^{(h)} + a_n^{(p)} = \alpha 2^n + n2^n = (\alpha + n)2^n$.
> *(Cannot show what the question wants; it is not correct.)*

b. [2pt] Use Theorem 5 to find all solutions of this recurrence relation.

> *All solutions by Theorem 5 are of the form* $a_n = (\alpha + n)2^n$, *as established above.*

c. [2pt] Find the solution with $a_0 = 2$.

> $(\alpha + 0)2^0 = \alpha = 2$. *Thus,* $\alpha = 2$ *and* $a_n = (2 + n)2^n$.