SC/MATH 1090

7- Boolean Semantics


York University
Department of Computer Science and Engineering
Overview

• Two main theorems:
  – Soundness: Our Boolean Logic is sound and truthful. Everything we can prove using the Boolean Logic is actually true.
  – Completeness: Our Boolean Logic is complete. Everything that is true (and can be represented in Boolean logic), the Boolean Logic can prove.
Soundness

• The primary rules of inference are truthful, i.e.

\[ A, A \equiv B \vdash_{taut} B \]

\[ A \equiv B \vdash_{taut} C[p := A] \equiv C[p := B] \]

• All logical axioms are tautologies.

• **Metatheorem. (Soundness of Propositional Calculus)**
  If \( \Gamma \vdash A \) then \( \Gamma \vdash_{taut} A \).
  – Proof by induction on length of \( \vdash \)-proofs where \( A \) occurs.

• **Corollary.** If \( \vdash A \), then \( \vdash_{taut} A \).
Counter-example construction

• **Soundness Theorem:**
  – If $\Gamma \vdash A$, then $\Gamma \models_{taut} A$.

• **Contrapositive** of Soundness theorem:
  – If $\Gamma \not\models_{taut} A$, then $\Gamma \not\models A$

• Reminder: $\Gamma \vdash A$ is a theorem schema.

• In order to show that $A$ is not provable, we can find a specific formula, and some state $v$ for which $v(A) = f$. 
Completeness

• **Metatheorem. (Post’s Tautology Theorem)**

\[ \text{If } \Gamma \models_{\text{taut}} A, \text{ then } \Gamma \vdash A. \]

• **Contrapositive** of Post theorem:

\[ \text{if } \Gamma \not\vdash A, \text{ then } \Gamma \not\models_{\text{taut}} A. \]