1- Boolean Formulae


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Overview

• Boolean syntax
  – Boolean Alphabet
  – Strings
  – Formula Calculation; well-formed-formula (WFF)
  – Parsing (top-down and bottom-up)
  – Removing redundant brackets
  – Complexity of formulae
Boolean Alphabet

1. Symbols for Boolean or propositional variables
   \( p, q, r \) with or without primes or subscripts
   Examples: \( p, p', p_{123}, q''_{45} \)

2. Symbols for Boolean constants
   \( \top \) called top, verum, or symbol “true”
   \( \bot \) called bottom, falsum, or symbol “false”

3. Brackets, ( and )

4. Boolean connectives
   \( \neg, \land, \lor, \rightarrow, \equiv \)
Strings or Expressions

• Definition:
  A **string** (or word, or expression) over a **given alphabet** is any **ordered** sequence of the alphabet’s symbols, written adjacent to each other without any visible separators (no commas or spaces, etc).

• Examples:
  – \((p \lor \bot)\) is a string given Boolean alphabet.
  – \((p\lnot q)\) is not a string given Boolean alphabet.
  – \((p \rightarrow q)\) and \(\rightarrow p)q(\) are two **different** strings given the Boolean alphabet. Note only the ordering is different.
Strings (cont.)

• String variables
  – Denoted by A, B, C, etc with or without primes or subscripts

• Concatenation
  – Example: if A is \textit{abc} and B is \textit{de} (given the English alphabet), then \textbf{AB} is \textit{abcde}

• Empty string
  – Denoted by \( \varepsilon \)
  – \( A\varepsilon = \varepsilon A = A \)

• Substring
  – “B is a substring of A” means that for some string C and D we have \( A = CBD \)
  – If B is a substring of A and \( B \neq A \), then B is a \textbf{proper substring} of A.
Formula calculation
Procedural definition

• Formula calculation is any finite (ordered) sequence of strings that we may write respecting the following requirements:

1. At any step, we may write a Boolean variable or a Boolean constant

2. At any step, we may write (¬ A), provided we have already written string A in a previous step.

3. At any step, we may write any of the strings (A ∧ B), (A ∨ B), (A → B), (A ≡ B) provided we have already written strings A and B in a previous step.
A string A over the Boolean alphabet is called a **Boolean Expression** or a **well-formed-formula** iff it is a string written at **some** step of some **formula-calculation**.

- Examples:
  \[(p \equiv q)\]
  \[((p \lor r) \rightarrow (\neg q))\]

- **WFF**: set of all well-formed-formulae (wffs)

- Bottom- up parsing of a wff is showing the procedural formula calculation steps.
Recursive definition of WFF

- The set of all well-formed-formulae is the smallest set of strings, WFF, that satisfies

  1. All Boolean variables (p, q, r, ...) , and constants (⊥, ⊤)
  2. If A and B are any strings in WFF, then so are the strings
     (¬ A), (A ∧ B), (A ∨ B), (A → B), (A ≡ B)

- Top-down parsing of a wff is showing the recursive formula calculation steps.

- How do we know recursion terminates?

- The two definitions for WFF are equivalent.
Immediate Predecessors (i.p.)

1. Boolean variables or constants don’t have any immediate predecessors

2. A is an immediate predecessor of (¬A)

3. A and B are immediate predecessors of (A ∧ B), (A ∨ B), (A → B), (A ≡ B)

• We will prove later that the i.p.s are unique for each formula.
Removing brackets

• Redundant brackets
  – Outermost brackets are redundant
  – Any pair of brackets is redundant if its presence can be understood from the priority of the connectives

• Priorities:
  – The order of priorities (decreasing) is agreed to be
    \( \neg, \land, \lor, \to, \equiv \)
  – For same connectives, the rightmost has the highest priority

• Least parenthesized notation (LPN): writing wff with all redundant brackets removed
  – Note writing wff in LPN is just a short notation and is not a correctly written formula (by formula calculation)
Complexity

• The **complexity** of a formula is the number of connectives occurring in the formula

• The complexity of Boolean variables and constants is zero (they are also called **atomic** formulae)

• Example
  - Complexity of \(((p \lor r) \rightarrow (\neg q))\) is 3