MATH/EECS 1019 Third test (version 2) – Fall 2014
Solutions

1. (3 points) If $x$ is an integer and 7 divides $3x+2$, prove that 7 also divides $15x^2-11x-14$.

   **Solution:**

   This is a direct proof. One can factorize the given expression, $15x^2-11x-14 = 5x(3x+2)-7(3x+2) = (3x+2)(5x-7)$, which shows that the given expression has $3x+2$ as a factor and thus must be divisible by 7.

2. (3 points) Prove that $p, p+2, p+4$ cannot all be primes except when $p=3$.
   Hint: Consider the different cases for $p$ mod 3 i.e. the remainder obtained after dividing $p$ by 3.

   **Solution:**

   As the hint suggests, we can do a proof by cases. Note that $p$ cannot be 3 (we know 3,5,7 are primes).

   Clearly $p$ mod 3 cannot be zero – if it is $p$ is a multiple of 3 and not 3 so $p$ cannot be prime. So there are 2 cases to consider.

   Case 1: $p$ mod $3=1$. In this case, $(p+2)$ mod $3=0$, i.e., $p+2$ is a multiple of 3 and thus not prime.

   Case 2: $p$ mod $3=2$. In this case, $(p+4)$ mod $3=0$, i.e., $p+4$ is a multiple of 3 and thus not prime.

   Thus in each case, $p, p+2, p+4$ cannot all be primes.

3. (3 points) Find the sum 1*3+2*4+3*5+...+99*101.

   **Solution:** The given sum is

$$
\begin{aligned}
S &= \sum_{i=1}^{99} i(i+2) \\
&= \sum_{i=1}^{99} (i^2+2i) \\
&= \sum_{i=1}^{99} i^2 + \sum_{i=1}^{99} 2i \\
&= \sum_{i=1}^{99} i^2 + 2\sum_{i=1}^{99} i \\
&= \frac{99*100*199}{6} + 2\left(\frac{99*100}{2}\right) \\
&= 33*50*199 + 99*100 \\
&= 338250
\end{aligned}
$$

4. (3 points) A fly starts at the origin and goes 1 unit up, 1/2 unit right, 1/4 unit down, 1/8 unit left, 1/16 unit up, etc., ad infinitum. In what coordinates does it end up?

   **Solution:** Let us compute the $x, y$ coordinates separately. The final $x$ coordinate is given by the infinite sum

   $$x = \frac{1}{2} - \frac{1}{8} + \frac{1}{32} - \frac{1}{128} + \dots$$

   This is a geometric series with $a = \frac{1}{2}$ and $r = -\frac{1}{4}$. So using the formula for the infinite geometric series we get $x = \frac{a}{1-r} = \frac{1/2}{1+1/4} = \frac{2}{5}$.

   Similarly, the final $y$ coordinate is given by the infinite sum

   $$y = \frac{1}{1} - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \dots$$

   This is a geometric series with $a = 1$ and $r = -\frac{1}{4}$. So using the formula for the infinite geometric series we get $y = \frac{a}{1-r} = \frac{1}{1+1/4} = \frac{4}{5}$.

5. (3 points) Prove, by induction on $n$, that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

   **Solution:** We prove this by using induction on $n$.

   Base Case: $n = 1$.
   True, since $1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36$ is divisible by 9.

   Inductive step: Assume the statement is true for $n = k$. So $k^3 + (k+1)^3 + (k+2)^3 = 9a$ for some natural number $a$. Then for $n = k+1$,

   $$\begin{aligned}(k+1)^3 + (k+2)^3 + (k+3)^3 &= (k+1)^3 + (k+2)^3 + (k+3)^3 + k^3 - k^3 \\ &= k^3 + (k+1)^3 + (k+2)^3 + (k+3)^3 - k^3 \\ &= 9a + (k+3)^3 - k^3 \\ &= 9a + 9k^2 + 27k + 27 \\ &= 9(a + k^2 + 3k + 3)\end{aligned}$$

   Hence by the principle of mathematical induction, the given statement is true.

6. (3 points) Let $k$ be a positive integer and $x$ be real. Prove using induction that if $x + \frac{1}{x}$ is an integer then $x^k + \frac{1}{x^k}$ is also an integer.

   **Solution:**

   We prove this by using strong induction on $k$.

   Base Case: $k = 1$. True, since $x + \frac{1}{x}$ is given to be an integer.

   Inductive step: Assume the statement is true for $k = m$ and $k = m-1$. So $x^{m-1} + \frac{1}{x^{m-1}}$ and $x^m + \frac{1}{x^m}$ are integers. Then for $k = m+1$,

   $$x^{m+1} + \frac{1}{x^{m+1}} = \left(x + \frac{1}{x}\right)\left(x^m + \frac{1}{x^m}\right) - \left(x^{m-1} + \frac{1}{x^{m-1}}\right)$$

   Since each of the three terms on the right hand side are integers, the left hand side is an integer. Hence by the principle of mathematical induction, the given statement is true.

7. (2+2 points) Define the factorial function of positive integers $m$ as $m! = 1 * 2 * \ldots * m$. Consider the question: "Prove that if $n > 4$ is composite, then $n$ divides $(n-1)!$." and the following "proof". What is the biggest flaw in the argument given in the proof? Fix the problem so that you get a correct proof.

   "Proof:" Since $n$ is composite, it has two factors $n_1, n_2$ that are each smaller than $n$. Since $(n-1)!$ is the product of all positive integers strictly smaller than $n$, this product must contain the numbers $n_1, n_2$, and therefore $(n-1)!$ is divisible by $n$.

   **Solution:** The proof breaks down when $n = k^2$ for some prime number $k$, so the only factorization it has gives two equal factors.

   The way to fix it is to handle this case separately. Since $n > 4$, $k > 2$. So $2k < k^2 = n$. and therefore $(n-1)!$ has factors $k, 2k$ that together is a multiple of $n$. Thus $(n-1)!$ is divisible by $n$ in this case as well.

8. (2 points) Suppose that $\log_4 x = y$ where $x$ is a positive real number. What is $\log_8 x$ in terms of $y$?

   **Solution:** Since $\log_4 x = y$, we have

$$\begin{aligned}
\log_4 x &= y \\
x &= 4^y \\
&= 2^{2y} \\
&= 2^{3\frac{2y}{3}} \\
&= (2^3)^{\frac{2y}{3}} \\
&= 8^{\frac{2y}{3}} \\
\log_8 x &= \frac{2y}{3}
\end{aligned}$$

9. (3 points) A function $f(x)$ is said to be strictly decreasing if $f(b) < f(a)$ for all $b > a$. Prove that a strictly decreasing function from $\mathbb{R}$ to itself is one-to-one.

   **Solution:** This is a very simple proof by contradiction. If $f$ is not one-to-one, there exists $x_1, x_2$ such that $f(x_1) = f(x_2)$. Without loss of generality, assume $x_1 < x_2$. Then $f(x_2) < f(x_1)$, which contradicts the assumption $f(x_1) = f(x_2)$.

10. (3 points) Consider the set of all fractions of the form $\frac{n}{n+\sqrt{n}}$, where $n \in \mathbb{Z}, n > 0$. Is the set countable? Prove your answer.

    **Solution:** This set is countable because we can define a bijection $f : \mathbb{N} \to \mathbb{R}, f(n) = \frac{n}{n+\sqrt{n}}$.

    Note that $f(n) = \frac{1}{1+1/\sqrt{n}}$. This mapping is one-to-one because if $f(n_1) = f(n_2)$ for some $n_1, n_2$ then $\frac{1}{1+1/\sqrt{n_1}} = \frac{1}{1+1/\sqrt{n_2}}$ which implies $n_1 = n_2$. The mapping $f$ is onto because every element in the given set is indexed by a natural number $n$. Any set that has a bijection from $\mathbb{N}$ is countable.