MATH/EECS 1019 Third test (version 1) – Fall 2014
Solutions

1. (3 points) Find the sum 1*2+2*3+3*4+...+99*100.

   Solution: The given sum is

$$
\begin{aligned}
S &= \sum_{i=1}^{99} i(i+1) \\
&= \sum_{i=1}^{99} (i^2+i) \\
&= \sum_{i=1}^{99} i^2 + \sum_{i=1}^{99} i \\
&= \frac{99*100*199}{6} + \frac{99*100}{2} \\
&= 33*50*199 + 99*50 \\
&= 333300
\end{aligned}
$$

2. (3 points) A fly starts at the origin and goes 1 unit up, 1/2 unit right, 1/4 unit down, 1/8 unit left, 1/16 unit up, etc., ad infinitum. In what coordinates does it end up?

   Solution: Let us compute the x, y coordinates separately. The final x coordinate is given by the infinite sum

$$x = \frac{1}{2} - \frac{1}{8} + \frac{1}{32} - \frac{1}{128} + \dots$$

   This is a geometric series with $a = \frac{1}{2}$ and $r = -\frac{1}{4}$. So using the formula for the infinite geometric series we get $x = \frac{a}{1-r} = \frac{1/2}{1+1/4} = \frac{2}{5}$.

   Similarly, the final y coordinate is given by the infinite sum

$$y = \frac{1}{1} - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \dots$$

   This is a geometric series with $a = 1$ and $r = -\frac{1}{4}$. So using the formula for the infinite geometric series we get $y = \frac{a}{1-r} = \frac{1}{1+1/4} = \frac{4}{5}$.

3. (3 points) If x is an integer and 7 divides $3x+2$, prove that 7 also divides $15x^2 - 11x - 14$.

   Solution:

   This is a direct proof. One can factorize the given expression, $15x^2 - 11x - 14 = 5x(3x + 2) - 7(3x+2) = (3x+2)(5x-7)$, which shows that the given expression has $3x+2$ as a factor and thus must be divisible by 7.

4. (3 points) Let $p < q$ be two consecutive odd primes. Prove that $p + q$ is a composite number, having at least three, not necessarily distinct, prime factors.

   Solution:

   This is another direct proof. Since p, q are *odd* primes, so $fracp + q2$ is an integer. Since p, q are *consecutive* primes and $p < \frac{p+q}{2} < q$, so $\frac{p+q}{2}$ is composite and must have at least two factors m, n. These facts imply that $2, m, n$ are factors of $p + q$.

5. (3 points) Prove, by induction on n, that

$$1 * 2 + 2 * 2^2 + 3 * 2^3 + \ldots + n * 2^n = 2 + (n - 1) * 2^{n+1}.$$

   Solution:

   We prove this by using induction on n.

   Base Case: $n = 1$. True, since $1 * 2 = 2 + (1 - 1) * 2^{0+1} = 2$.

   Inductive step: Assume the statement is true for $n = k$. So

$$1 * 2 + 2 * 2^2 + 3 * 2^3 + \ldots + m * 2^m = 2 + (m - 1) * 2^{m+1}.$$

   Then for $k = m + 1$,

$$\begin{aligned} & 1 * 2 + 2 * 2^2 + 3 * 2^3 + \ldots + m * 2^m + (m + 1) * 2^{m+1} \\ = \; & 2 + (m - 1) * 2^{m+1} + (m + 1) * 2^{m+1} \\ = \; & 2 + [(m - 1) + (m + 1)] * 2^{m+1} \\ = \; & 2 + 2m * 2^{m+1} \\ = \; & 2 + m * 2^{m+2} \end{aligned}$$

   Hence by the principle of mathematical induction, the given statement is true.

6. (3 points) Prove using mathematical induction that if n non-parallel straight lines on the plane intersect at a common point, they divide the plane into $2n$ regions.

   Solution:

   We prove this by using induction on n.

   Base Case: $n = 1$. True, since one straight line divides the plane into 2 regions.

   Inductive step: Suppose that the hypothesis is true for $n = m$, so that any m non-parallel straight lines on the plane intersecting at a common point divide the plane into $2m$ regions. Then for $n = m + 1$, we have to show that any $m + 1$ non-parallel straight lines on the plane intersecting at a common point divide the plane into $2m + 2$ regions.

   Choose any set of m lines from a given set of $m + 1$ non-parallel straight lines on the plane intersecting at a common point. By the inductive hypothesis, these lines divide the plane into $2m$ regions. Since the $m + 1^{th}$ line also passes through the common intersection, it passes through exactly two of the $2m$ regions. Since it cuts each of these two regions in two parts, it creates a total of $2m+2$ regions.

Hence by the principle of mathematical induction, the given statement is true.

7. (3 points) A function $f(x)$ is said to be strictly increasing if $f(b) > f(a)$ for all $b > a$. Prove that a strictly increasing function from $\mathbb{R}$ to itself is one-to-one.

   Solution: This is a very simple proof by contradiction. If f is not one-to-one, there exists x_1, x_2 such that $f(x_1) = f(x_2)$. Without loss of generality, assume $x_1 < x_2$. Then $f(x_1) < f(x_2)$, which contradicts the assumption $f(x_1) = f(x_2)$.

8. (2 points) Suppose that $\log_4 x = y$ where x is a positive real number. What is $\log_{16} x$ in terms of y?

   Solution: Since $\log_4 x = y$, we have

$$
\begin{aligned}
\log_4 x &= y \\
x &= 4^y \\
&= 2^{2y} \\
&= 2^{4\frac{2y}{4}} \\
&= (2^4)^{\frac{y}{2}} \\
&= 16^{\frac{y}{2}} \\
\log_{16} x &= \frac{y}{2}
\end{aligned}
$$

9. (2+2 points) Define the factorial function of positive integers m as $m! = 1 * 2 * \ldots * m$. Consider the question: “Prove that if $n > 4$ is composite, then n divides $(n-1)!$.” and the following “proof”. What is the biggest flaw in the argument given in the proof? Fix the problem so that you get a correct proof.

   “Proof:” Since n is composite, it has two factors n_1, n_2 that are each smaller than n. Since $(n-1)!$ is the product of all positive integers strictly smaller than n, this product must contain the numbers n_1, n_2, and therefore $(n-1)!$ is divisible by n.

   Solution: The proof breaks down when $n = k^2$ for some prime number k, so the only factorization it has gives two equal factors.

   The way to fix it is to handle this case separately. Since $n > 4$, $k > 2$. So $2k < k^2 = n$. and therefore $(n-1)!$ has factors $k, 2k$ that together is a multiple of n. Thus $(n-1)!$ is divisible by n in this case as well.

10. (3 points) Consider the set of all fractions of the form $\frac{n}{n+\sqrt{2}}$, where $n \in \mathbb{Z}$. Is the set countable? Prove your answer.

    Solution: Consider first the set of all fractions of the form $\frac{n}{n+\sqrt{2}}$, where $n \in \mathbb{N}$. This set is countable because we can define a mapping $f : \mathbb{N} \to \mathbb{R}, f(n) = \frac{n}{n+\sqrt{2}}$.

Note that $f(n) = \frac{1}{1+\sqrt{2}/n}$. This mapping is one-to-one because if $f(n_1) = f(n_2)$ for some n_1, n_2 then $\frac{1}{1+\sqrt{2}/n_1} = \frac{1}{1+\sqrt{2}/n_2}$ which implies $n_1 = n_2$. It is onto because every number in the given set is indexed by a natual number n. Any set that has a bijection from $\mathbb{N}$ is countable.

Now we have to modify the mapping to work for the set given in the question. We can do that the usual way, as follows. Define a mapping g : as follows.

$g : \mathbb{N} \rightarrow \mathbb{R}$ as follows,

$$\begin{aligned} g(n) &= 0 \text{ if } n = 0 \\ &= f(m) \text{ if } n = 2m + 1 \\ &= f(-m) \text{ if } n = 2m \end{aligned}$$

g is one-to-one because f is one-to-one and thus provides a one-to-one mapping from $\mathbb{N}$ to the set given. g is onto because every number in the set given is indexed with an integer and g provides a bijection from the integers to the natural numbers. Therefore the given set is countable.