MATH/EECS 1019: Discrete Math for Computer Science
Fall 2014 – Solutions

Question 1

[5 points] Prove that $8n^3 + \sqrt{n} \in \Theta(n^3)$.

Solution: First let us prove the $\Omega()$ part. Since $8n^3 + \sqrt{n} > 8n^3$ for $n \geq 1$, we can use $c = 8, n_0 = 1$ in the definition of $\Omega()$.

Next we prove the $O()$ part. Since $8n^3 + \sqrt{n} \leq 8n^3 + n^3 = 9n^3$ for $n \geq 1$, we can use $c = 9, n_0 = 1$ in the definition of $O()$. This completes the proof.

Question 2

[5 points] What is the value returned by the following function? Express your answer as a function of n. Give using $O()$ notation the worst-case running time.

F1(n)
1 $v \leftarrow 0$
2 **for** $i \leftarrow 1$ **to** n
3 **do for** $j \leftarrow n+1$ **to** $2n$
4   **do** $v \leftarrow v + 1$
5 **return** v

Solution: Line 4 runs exactly n^2 times and adds 1 to v each time it executes. Therefore the final value of v, which is returned, is n^2.

For the running time you can do a detailed analysis as done in the class or you can argue that because it has two nested loops, each running n iterations, so the running time is $O(n^2)$.

Question 3

[5 points] Argue informally that the following recursive algorithm is correct. The algorithm (called as rmax(1,n)) finds the maximum of a list of numbers contained in an array $S[1..n]$.

RMAX(x, y)
// return maximum in $S[x..y]$
if $y - x \leq 1$
  then return $maximum(S[x], S[y])$
  else $max_1 \leftarrow \text{RMAX}(x, \lfloor \frac{x+y}{2} \rfloor)$
      $max_2 \leftarrow \text{RMAX}(\lfloor \frac{x+y}{2} \rfloor + 1, y)$
      return $maximum(max_1, max_2)$

Solution: If the array has a single element, the function returns that element, which is correct, since it is the maximum. If it has more, the lines 4,5 break the array into two roughly equal parts and then (recursively) computes maximum of each part. Line 6 computes the maximum of these two numbers. This is the maximum of the array since the larger of the maximums of each part must be the largest number in the array.

Question 4

[5 points] Prove that $\log n! \in O(n \log n)$.

Solution: Note that

$$\log n! = \sum_{j=1}^{n} \log j.$$

Also $\log j \leq \log n$ if $1 \leq j \leq n$. So we have

$$\sum_{j=1}^{n} \log j \leq \sum_{j=1}^{n} \log n = n \log n$$

Therefore $\log n! \leq n \log n$ for $n \in \mathbb{N}$. Thus we can use $c = 1$ and $n_0 = 1$ in the definition of $O()$ and show that $\log n! \in O(n \log n)$.