MATH/EECS 1019: Discrete Math for Computer Science
Fall 2014
Assignment 3 – Solutions

Question 1

[2+2 points] Functions

1. Find $g^{-1}(3)$ given $g(x) = \frac{3x+1}{2x+g(x)}$.

   Solution: We have to find the value of x such that $g(x) = 3$. So we can substitute $g(x) = 3$ and solve for x. Therefore, we have

$$\begin{aligned} g(x) &= \frac{3x+1}{2x+g(x)} \\ 3 &= \frac{3x+1}{2x+3} \\ 3(2x+3) &= 3x+1 \\ 3x &= -8 \\ x &= -8/3 \end{aligned}$$

2. Let f be the function $f(x) = ax^2 - \sqrt{2}$ for some positive real number a. If $f(f(\sqrt{2})) = -\sqrt{2}$ what is a?

   Solution:

$$\begin{aligned} f(\sqrt{2}) &= 2a - \sqrt{2} \\ f(f(\sqrt{2})) &= a(2a - \sqrt{2})^2 - \sqrt{2} \\ a(2a - \sqrt{2})^2 - \sqrt{2} &= -\sqrt{2} \\ a(2a - \sqrt{2})^2 &= 0 \\ (2a - \sqrt{2})^2 &= 0 \\ 2a &= \sqrt{2} \\ a &= 1/\sqrt{2} \end{aligned}$$

Question 2

[2+2 points] Logarithms

1. Let $x = 2^{\log_b 3}$ and $y = 3^{\log_b 2}$. Find $x - y$.

   Solution: Note that

$$\begin{aligned}
\log_b x &= \log_b 2^{\log_b 3} \\
&= \log_b 3 \log_b 2 \text{ and} \\
\log_b y &= \log_b 3^{\log_b 2} \\
&= \log_b 2 \log_b 3
\end{aligned}$$

So $\log_b x = \log_b y$ and therefore $x = y$ and thus $x - y = 0$.

2. If $\frac{\log_b a}{\log_c a} = \frac{19}{99}$ and $\frac{b}{c} = c^k$, compute k.

   Solution: Let us change the logarithm base to c in the numerator. Note that $\log_b a = \frac{\log_c a}{\log_c b}$. So

$$\begin{aligned}
\frac{\log_b a}{\log_c a} &= \frac{\log_c a}{\log_c a \log_c b} \\
&= \frac{1}{\log_c b} \\
&= \frac{19}{99}
\end{aligned}$$

   So we have

$$\begin{aligned}
\log_c b &= \frac{99}{19} \\
b &= c^{\frac{99}{19}} \\
\frac{b}{c} &= c^{\frac{80}{19}}
\end{aligned}$$

   Thus $k = \frac{80}{19}$.

Question 3

[4 points] Sequences and Series

The first four terms of an arithmetic sequence (in order) are $x + y, x - y, xy$ and x/y. What is the value of the fifth term?

Solution: Since this is an arithmetic sequence, every pair of adjacent terms must have the same common difference. The first pair has the common difference $-2y$. We also have

$$\begin{aligned}
xy - (x - y) &= -2y \\
xy - x &= -3y \\
x &= \frac{-3y}{y - 1}
\end{aligned}$$

Finally

$$\begin{aligned} x/y - xy &= -2y \\ x - xy^2 &= -2y^2 \\ x &= \frac{-2y^2}{1-y^2} \end{aligned}$$

So we have

$$\frac{-3y}{y-1} = \frac{-2y^2}{1-y^2}.$$

Note that $y = 0$ is not possible (x/y must be defined) and $y = 1$ is not possible (the first two terms would be unequal but the last two terms would be equal). So we can cancel y from the numerators and $1 - y$ from the denominators.

So we have

$$\begin{aligned} \frac{-3y}{y-1} &= \frac{-2y^2}{1-y^2} \\ \frac{-3}{y-1} &= \frac{-2y}{1-y^2} \\ \frac{-3}{-1} &= \frac{-2y}{1+y} \\ 3+3y &= -2y \\ 5y &= -3 \\ y &= -3/5 \\ x &= \frac{-3y}{y-1} \\ &= \frac{9/5}{-8/5} \\ &= -\frac{9}{8} \end{aligned}$$

So the fifth term is $x/y - 2y = 15/8 + 6/5 = 3.075$.

Question 4

[4 points] Sequences and Series

Find a formula, in terms of n, for the sum of the first n terms of the sequence

$$1, 1+2, 1+2+2^2, 1+2+2^2+2^3, ...$$

Solution: Let us find a formula for the i^{th} term t_i, $i = 1, 2, ...$ From inspection,

$$\begin{aligned} t_i &= \sum_{j=0}^{i-1} 2^j \\ &= \frac{2^i - 1}{2 - 1} \\ &= 2^i - 1 \end{aligned}$$

Therefore the sum we have to compute is

$$\begin{aligned} S_n &= \sum_{i=1}^{n} t_i \\ &= \sum_{i=1}^{n} (2^i - 1) \\ &= \sum_{i=1}^{n} 2^i - \sum_{i=1}^{n} 1 \\ &= \frac{2(2^n - 1)}{2 - 1} - n \\ &= 2^{n+1} - n - 2 \end{aligned}$$

Question 5

[4 points] Functions, Induction

Let $f : \mathbb{R} \to \mathbb{R}, f(x) = \frac{x}{1-x}$. Define $f^2(x) = f(f(x)), f^3(x) = f(f(f(x)))$ and so on. Guess the form for $f^n(x)$ and prove your answer correct using induction on n.

Solution: We can check that

$$\begin{aligned} f(f(x)) &= \frac{f(x)}{1 - f(x)} \\ &= \frac{\frac{x}{1-x}}{1 - \frac{x}{1-x}} \\ &= \frac{x}{1 - 2x} \end{aligned}$$

We can continue and check that $f(f(f(x))) = \frac{x}{1-3x}$ and so on and can conjecture that $f^n(x) = \frac{x}{1-nx}$. Let us prove this using induction on n.

Base Case: For $n = 1$ the definition of $f(x)$ satisfies the hypothesis.

Inductive Step: Suppose the statement is true for $n = k$ so that $f^k(x) = \frac{x}{1-kx}$. We need to prove that the statement holds for $n = k+1$, i.e., $f^{k+1}(x) = \frac{x}{1-(k+1)x}$.

$$
\begin{aligned}
f^{k+1}(x) &= \frac{f^k(x)}{1 - f^k(x)} \\
&= \frac{\frac{x}{1-kx}}{1 - \frac{x}{1-kx}} \\
&= \frac{\frac{x}{1-kx}}{\frac{1-(k+1)x}{1-kx}} \\
&= \frac{x}{1-(k+1)x}
\end{aligned}
$$

Thus the conjecture is true.