QUESTION 1. (3 MARKS) Use truth table shortcuts to show

\[ A \models \text{taut } B \overset{\text{iff}}{=} \models \text{taut } A \rightarrow B \]

Note iff!

Answer. Two proofs:

(1) If \( A \models \text{taut } B \), then \( \models \text{taut } A \rightarrow B \)
Assume that \( A \rightarrow B \) is not a tautology. Then there must be a state \( v \) for which \( v(A \rightarrow B) = f \). This is only possible if \( v(A) = t \) and \( v(B) = f \). But this contradicts the assumption \( A \models \text{taut } B \) by which for all states that make \( A \) true, we have \( B \) true as well. Since assuming that \( 'A \rightarrow B \) is not a tautology’ resulted in a contradiction, it must be wrong and indeed \( \models \text{taut } A \rightarrow B \) holds (is a tautology).

(2) If \( \models \text{taut } A \rightarrow B \), then \( A \models \text{taut } B \)
Assume that \( A \models \text{taut } B \) is not a valid tautological implication. Therefore there exists a state \( v \) for which \( v(A) = t \) but \( v(B) = f \). For such a state \( v(A \rightarrow B) = f \). This contradicts our assumption that \( \models \text{taut } A \rightarrow B \), which means \( A \rightarrow B \) must be true for all states. Since assuming that \( 'A \models \text{taut } B \) is not a valid tautological implication’ resulted in a contradiction, it must be incorrect and indeed \( A \models \text{taut } B \) holds is a tautological implication.

QUESTION 2. (a) (2 MARKS) Give without comment a formula calculation for

\[ (((\neg r) \equiv \top) \equiv (q \lor (\perp \rightarrow p))) \]

(b) (1 MARK) Give without comment the least-parenthesized form for the above.

Answer.
(a) Formula calculation:
\[ r, (\neg r), \top, ((\neg r) \equiv \top), q, \perp, p, (\perp \rightarrow p), (q \lor (\perp \rightarrow p)), (((\neg r) \equiv \top) \equiv (q \lor (\perp \rightarrow p))) \]

(b) Least-parenthesized form: \( (\neg r \equiv \top) \equiv q \lor (\perp \rightarrow p) \)

QUESTION 3. (4 MARKS) Is \( \top \) a formula? Prove your answer.

Answer. \( \top \) is NOT a formula.
A string written in Boolean alphabet is a formula if it can be written as a step of some formula calculation.
(1) The above string cannot be written using the first rule, since only Boolean variables or constants can be written by rule 1 and they do not have any brackets.
(2) The above string cannot be written using the second rule, since it does not contain \( \neg \).
(3) The above string cannot be written using the third rule, since it does not contain \( \land, \lor, \rightarrow, \) or \( \equiv \). Therefore \( \top \) is not a well-formed formula.
QUESTION 4. (4 MARKS) Write a Hilbert-style proof for
\[ A, B \vdash (A \rightarrow B) \land (B \rightarrow A) \]
Use of deduction theorem is NOT allowed.

Answer.

(1) \( A \) <assumption >
(2) \( B \) <assumption >
(3) \( A \lor \neg B \) <(1) + A \vdash A \lor X >
(4) \( B \rightarrow A \) <(3)+ implication theorem+Eqn >
(5) \( B \lor \neg A \) <(2) + B \vdash B \lor X >
(6) \( A \rightarrow B \) <(5)+ implication theorem+Eqn >
(7) \( (A \rightarrow B) \land (B \rightarrow A) \) <(4)+(6)+merge >

QUESTION 5. Write an Equational-style proof for each of the following. Do NOT use the deduction theorem.

Answer.

(a) (4 MARKS) \( A \lor B, \neg A \vdash B \)

\[
A \lor B \\
\Leftrightarrow < \text{Double negation+Leib, C-part: } p \lor B, \text{p fresh } > \\
\neg \neg A \lor B \\
\Leftrightarrow < \text{assumption } \neg A+ \text{redundant true metatheorem+ leib, C-part: } \neg p \lor B, \text{p fresh } > \\
\neg \top \lor B \\
\Leftrightarrow < \text{proven theorem } \vdash \top \equiv \neg \bot > \\
\neg \neg \bot \lor B \\
\Leftrightarrow < \text{Double negation+Leib, C-part: } p \lor B, \text{p fresh } > \\
\bot \lor B \\
\Leftrightarrow < \text{proven theorem } \vdash \bot \lor B \equiv B > \\
B
\]

(b) (4 MARKS) \( \bot \vdash A \)

\[
A \\
\Leftrightarrow < \text{proven theorem } \vdash A \lor \bot \equiv A > \\
A \lor \bot \\
\Leftrightarrow < \text{assumption } \bot+ \text{redundant true metatheorem+ leib, C-part: } A \lor p, \text{p fresh } > \\
A \lor \top 
\]

Bingo! This is a proven theorem.
QUESTION 6. (4 MARKS) Prove by resolution

\[ \vdash (A \to C) \to (B \to C) \to (A \lor B \to C) \]

**Answer.** This was done in the class!

QUESTION 7. (2 MARKS) For each of the following statements, indicate whether it is correct or incorrect. If incorrect, correct the statement by re-writing it.

(a) Deduction theorem says that if \( \Gamma \vdash A \to B \), then \( \Gamma + A \vdash B \).

**Answer.** This is INCORRECT. The Deduction theorem says that if \( \Gamma + A \vdash B \), then \( \Gamma \vdash A \to B \).

(b) In a Hilbert-style proof for \( \Gamma \vdash B \), we can start by writing \( B \) on the first line of proof and show it is equivalent to an axiom, an assumption, or a proven theorem on the last line of proof.

**Answer.** This is INCORRECT. This can be done in an Equational proof. In a Hilbert proof, we can start by writing an axiom, an assumption, or a proven theorem on the first line of proof and show \( B \) on the last line of proof, or earlier.

QUESTION 8. (2 MARKS) Which of the following substitutions are legal? If illegal, explain why. If legal, give the answer after applying substitution.

(a) \( p \land \bot \left[ \bot \vDash r \right] \)

**Answer.** This is ILLEGAL. In a legal substitution, we can substitute for a Boolean variable, and \( \bot \) is not a Boolean variable.

(b) \( \neg q \lor p \left[ q \vDash p \right] \)

**Answer.** The substitution is legal and results in \( \neg q \lor p \), since the priority of substitution is higher than \( \lor \).