1. (3 marks) (Question 8, section 6.6 of book) Prove if x is not free in A,
\[ \vdash A \lor (\forall x)B \lor C \equiv (\forall x)_B(A \lor C) \]
\[ A \lor (\forall x)B \lor C \]
\[ \Leftrightarrow < \text{Translating to standard notation} > \]
\[ A \lor (\forall x)(B \rightarrow C) \]
\[ \Leftrightarrow < 6.4.2 \text{ given } x \text{ donf in } A > \]
\[ (\forall x)(A \lor (B \rightarrow C)) \]
\[ \Leftrightarrow < \text{Implication theorem + WL, C-part: } (\forall x)(A \lor p), p \text{ fresh } > \]
\[ (\forall x)(A \lor \neg B \lor C) \]
\[ \Leftrightarrow < \text{ Symmetry of } \lor + \text{ Implication theorem + WL, C-part: } (\forall x)p, p \text{ fresh } > \]
\[ (\forall x)(B \rightarrow (A \lor C)) \]
\[ \Leftrightarrow < \text{ Translating to subscript notation } > \]
\[ (\forall x)_B(A \lor C) \]

2. (3 marks) (Question 12, section 6.6 of book) Prove the \textit{one point rule- }\exists \text{ version:}
\[ (\exists x)(x = t \land A) \equiv A[x := t], \text{ if } x \text{ is not free in } t \]
\[ (\exists x)(x = t \land A) \]
\[ \Leftrightarrow < \text{Definition of } \exists > \]
\[ \neg(\forall x)\neg(x = t \land A) \]
\[ \Leftrightarrow < \text{ De Morgan + WL, C-part: } \neg(\forall x)p, p \text{ fresh } > \]
\[ \neg(\forall x)(\neg x = t \lor \neg A) \]
\[ \Leftrightarrow < \text{Implication theorem + WL, C-part: } \neg(\forall x)p, p \text{ fresh } > \]
\[ \neg(\forall x)(x = t \rightarrow \neg A) \]
\[ \Leftrightarrow < \text{ One point rule, given } x \text{ donf in } t + \text{ SL, C-part: } \neg p, p \text{ fresh } > \]
\[ \neg(\neg A)[x := t] \]
\[ \Leftrightarrow < \text{Definition of substitution + double negation } > \]
\[ A[x := t] \]

Note that I could use WL instead of SL above, since I am using the \textit{one point rule} which is an absolute theorem.
3. (3 marks) (Question 14, section 6.6 of book) Prove the dual of Range Split:

\[(\exists x)_{A \lor B} C \equiv (\exists x)_A C \lor (\exists x)_B C\]

\[(\exists x)_{A \lor B} C\]
\[\iff \text{ Translating to standard notation }\]
\[\iff (\exists x)((A \lor B) \land C)\]
\[\iff \text{ Definition of } \exists\]
\[-(\forall x)\neg((A \lor B) \land C)\]
\[\iff \text{ De Morgan + WL, C-part: } -(\forall x)p, p \text{ fresh }\]
\[-(\forall x)(\neg(A \lor B) \lor \neg C)\]
\[\iff \text{ Implication theorem + WL, C-part: } -(\forall x)p, p \text{ fresh }\]
\[-(\forall x)((A \lor B) \rightarrow \neg C)\]
\[\iff \text{ Range split + SL, C-part: } -p, p \text{ fresh }\]
\[-(\forall x)((\forall x)(A \rightarrow \neg C) \land (\forall x)(B \rightarrow \neg C))\]
\[\iff \text{ De Morgan }\]
\[-(\forall x)(A \rightarrow \neg C) \lor -(\forall x)(B \rightarrow \neg C)\]
\[\iff \text{ Implication theorem + de Morgan + WL twice }\]
\[-(\forall x)\neg(A \land C) \lor -(\forall x)\neg(B \land C)\]
\[\iff \text{ Definition of } \exists \text{ twice }\]
\[(\exists x)(A \land C) \lor (\exists x)(B \land C)\]
\[\iff \text{ Translating to subscript notation }\]
\[(\exists x)_A C \lor (\exists x)_B C\]

Note that I could use WL instead of SL above, since I am using the range split rule which is an absolute theorem.

4. (3 marks) (Question 17, section 6.6 of book) In which step, did the following proof for \(\vdash (\forall x)(\exists y)A \rightarrow (\exists y)(\forall x)A\) go wrong:

(1) \((\forall x)(\exists y)A\) \(\text{ <hypothesis >}\)
(2) \((\exists y)A\) \(\text{ <(1) + spec >}\)
(3) \(A[y := z]\) \(\text{ <auxiliary hypothesis associated with (2), z fresh >}\)
(4) \((\forall x)A[y := z]\) \(\text{ <(3) + gen, xdnof in line(1) >}\)
(5) \((\exists y)(\forall x)A\) \(\text{ <(4) + dual of spec >}\)

Line (4) is wrong! Weak generalization with \((\forall x)\) can be applied if x dnof in any hypothesis. However line (3), which is also a hypothesis, is not checked for free x, and
in fact, it might have free x. Therefore the generalization is invalid and results in a wrong proof.

Note that the proof is using deduction theorem to prove instead \((\forall x)(\exists y)A \vdash (\exists y)(\forall x)A\).

5. (3 marks) (Question 18, section 6.6 of book) Use the Auxiliary Variable Metatheorem to prove \(\vdash (\exists x)(A \rightarrow B) \rightarrow (\forall x)A \rightarrow (\exists x)B\).

By deduction theorem, it is sufficient to prove instead 
\[(\exists x)(A \rightarrow B), (\forall x)A \vdash (\exists x)B\]

\[(1) \quad (\exists x)(A \rightarrow B) \quad \langle \text{hypothesis} \rangle\]
\[(2) \quad (\forall x)A \quad \langle \text{hypothesis} \rangle\]
\[(3) \quad (A \rightarrow B)[x := z] \quad \langle \text{auxiliary hypothesis associated with (1), z fresh} \rangle\]
\[(4) \quad A[x := z] \rightarrow B[x := z] \quad \langle (3) + \text{substitution} \rangle\]
\[(5) \quad A[x := z] \quad \langle (2) + \text{spec} \rangle\]
\[(6) \quad B[x := z] \quad \langle (4) + (5) + \text{MP} \rangle\]
\[(7) \quad (\exists x)B \quad \langle (6) + \text{dual of spec} \rangle\]

6. (3 marks) (Question 20, section 6.6 of book) Use the Auxiliary Variable Metatheorem to prove \(\vdash (\exists x)(A \land C) \rightarrow (\exists x)(A \lor B) \land C\).

In standard notation: \(\vdash (\exists x)(A \land C) \rightarrow (\exists x)((A \lor B) \land C)\)

By deduction theorem, it is sufficient to prove instead 
\[(\exists x)(A \land C) \vdash (\exists x)((A \lor B) \land C)\]

\[(1) \quad (\exists x)(A \land C) \quad \langle \text{hypothesis} \rangle\]
\[(2) \quad A[x := z] \land C[x := z] \quad \langle \text{auxiliary hypothesis associated with (1), z fresh} \rangle\]
\[(3) \quad A[x := z] \quad \langle (2) + \text{split} \rangle\]
\[(4) \quad C[x := z] \quad \langle (2) + \text{split} \rangle\]
\[(5) \quad A[x := z] \lor B[x := z] \quad \langle (3) + 2.5.1(3) \rangle\]
\[(6) \quad (A[x := z] \lor B[x := z]) \land C[x := z] \quad \langle (4) + (5) + \text{merge} \rangle\]
\[(7) \quad ((A \lor B) \land C)[x := z] \quad \langle (6) + \text{substitution} \rangle\]
\[(8) \quad (\exists x)((A \lor B) \land C) \quad \langle (7) + \text{dual of spec} \rangle\]
7. (3 marks) (Question 22, section 6.6 of book) Prove \((\exists x)(A \rightarrow (\forall x)A)\).

By proof by contradiction, we can instead prove \(\neg(\exists x)(A \rightarrow (\forall x)A) \vdash \bot\)

\[
\begin{align*}
(1) & \quad \neg(\exists x)(A \rightarrow (\forall x)A) \quad \text{<hypothesis>} \\
(2) & \quad \neg(\forall x)\neg(A \rightarrow (\forall x)A) \quad \text{<1> + definition of \(\exists\) + WL, C-part: \(\neg p\), p fresh + Eqn} \\
(3) & \quad (\forall x)\neg(A \rightarrow (\forall x)A) \quad \text{<2> + double negation + Eqn} \\
(4) & \quad \neg(A \rightarrow (\forall x)A) \quad \text{<3> + spec} \\
(5) & \quad A \land \neg(\forall x)A \quad \text{<4> + tautological implication} \\
(6) & \quad A \quad \text{<5> + split} \\
(7) & \quad \neg(\forall x)A \quad \text{<5> + split} \\
(8) & \quad (\forall x)A \quad \text{<6> + gen, x donf in line (1)} \\
(9) & \quad \bot \quad \text{<7> + (8) + cut rule}
\end{align*}
\]

Here is another proof:

\[(\exists x)(A \rightarrow (\forall x)A) \]
\[
\iff \quad < \text{Definition of } \exists > \\
\quad \neg(\forall x)\neg(A \rightarrow (\forall x)A) \\
\iff \quad < \text{Implication theorem + WL, C-part: } \neg(\forall x)\neg p, p \text{ fresh} > \\
\quad \neg(\forall x)\neg (\neg A \lor (\forall x)A) \\
\iff \quad < \text{De Morgan + WL, C-part: } \neg(\forall x)p, p \text{ fresh} > \\
\quad (\forall x)(A \land \neg(\forall x)A) \\
\iff \quad < \text{Distributivity of } \forall \text{ over } \land + SL, C-part: \neg p, p \text{ fresh} > \\
\quad \neg((\forall x)A \land (\forall x)\neg(\forall x)A) \\
\iff \quad < \text{De morgan} > \\
\quad \neg(\forall x)A \lor \neg(\forall x)\neg(\forall x)A \\
\iff \quad < \text{x donf in } \neg(\forall x)A, \text{ therefore } \vdash \neg(\forall x)A \equiv (\forall x)\neg(\forall x)A + WL, C-part: \neg(\forall x)A \lor \neg p, p \text{ f.} > \\
\quad \neg(\forall x)A \lor \neg\neg(\forall x)A \\
\iff \quad < \text{Double negation + WL, C-part: } \neg(\forall x)A \lor p, p \text{ fresh} > \\
\quad \neg(\forall x)A \lor (\forall x)A
\]

Bingo! This is the excluded middle axiom (given symmetry of \(\lor\)).

Note in the above proof, I used the \(\vdash A \equiv (\forall x)A\) provided x donf in A - which we proved in the class using axiom 2 and axiom 4.
8. (3 marks) (Question 26, section 6.6 of book) Prove if x is not free in B,

\[ \vdash (\exists x)A \rightarrow ((\exists x)_A(B \lor C)) \equiv B \lor (\exists x)_A(C) \]

In standard notation: \[ \vdash (\exists x)A \rightarrow ((\exists x)(A \land (B \lor C))) \equiv B \lor (\exists x)(A \land C) \]

By deduction theorem, it is sufficient to prove instead \[ (\exists x)A \vdash (\exists x)(A \land (B \lor C)) \equiv B \lor (\exists x)(A \land C) \]

\[ (\exists x)(A \land (B \lor C)) \]
\[ \Leftrightarrow < \text{Distributivity of } \exists \text{ over } \lor \text{ (result 5 page 175 of book) } > \]
\[ (\exists x)(A \land B) \lor (\exists x)(A \land C) \]
\[ \Leftrightarrow < 6.4.3 \text{ given x dnof in B, WL, C-part: } p \lor (\exists x)(A \land C), p \text{ fresh } > \]
\[ (B \land (\exists x)A) \lor (\exists x)(A \land C) \]
\[ \Leftrightarrow < \text{assumption } ((\exists x)A) + \text{red. true metathm + SL, C-part: } (B \land p) \lor (\exists x)(A \land C), p \text{ fresh } > \]
\[ (B \land \top) \lor (\exists x)(A \land C) \]
\[ \Leftrightarrow < 2.4.20 + WL, C-part: p \lor (\exists x)(A \land C), p \text{ fresh } > \]
\[ B \lor (\exists x)(A \land C) \]

9. (3 marks) (Question 1, section 8.3 of book)

(a) Prove by an appropriate countermodel argument that

\[ (((\forall x)A \rightarrow (\forall x)B) \rightarrow (\forall x)(A \rightarrow B)) \]

is not a universally valid schema.

(b) Using your answer in part (a), show that the above formula cannot be a theorem schema.

(a) Let’s assume \( D = \mathbb{N} \) is the set of natural numbers, \( A \) is \( x < y \), \( B \) is \( x < z \), \( y^P = 5 \), \( z^P = 3 \), then

\( A^D \) is \( ((\forall x \in \mathbb{N})x < 5 \rightarrow (\forall x \in \mathbb{N})x < 3) \rightarrow (\forall x \in \mathbb{N})(x < 5 \rightarrow x < 3) \)

We can see that \( (\forall x \in \mathbb{N})x < 5 \) is \( f \). Take \( x \) to be 6 for example. Then \( x < 5 \) is \( f \), therefore \( (\forall x \in \mathbb{N})x < 5 \) is \( f \).

For similar reason, \( (\forall x \in \mathbb{N})x < 3 \) is also \( f \).

Take \( x \) to be 4. We can see that \( x < 5 \rightarrow x < 3 \) is \( f \), therefore \( (\forall x \in \mathbb{N})(x < 5 \rightarrow x < 3) \) is \( f \).

Thus \( A^D \) translates to \( (f \rightarrow f) \rightarrow f \) which is \( f \).

Therefore this interpretation is a counter model of \( A \) or \( \not\vDash_D ((\forall x)A \rightarrow (\forall x)B) \rightarrow (\forall x)(A \rightarrow B) \). Therefore the above formula is not universally valid, which can be shown as

\[ \not\vDash ((\forall x)A \rightarrow (\forall x)B) \rightarrow (\forall x)(A \rightarrow B) \]
(b) Based on contrapositive of soundness theorem in first order logic, since
\[ \not\models (\forall x)A \rightarrow (\forall x)B \rightarrow (\forall x)(A \rightarrow B), \]
then \[ \not\models (\forall x)A \rightarrow (\forall x)B \rightarrow (\forall x)(A \rightarrow B). \]

10. (3 marks) (Question 2, section 8.3 of book) Is the following schema a derived rule of our logic?

\[ A \rightarrow B \vdash A \rightarrow (\forall x)B, \text{ provided } x \text{ is not free in } A \]

If yes, then give a proof. If no, show why by proving the invalid "strong generalization" using the above formula.

The above rule is NOT a derived rule, since it can be used to prove strong generalization, i.e. \( A \vdash (\forall x)A \) as follows:

\[
\begin{align*}
(1) & : A & <\text{hypothesis} > \\
(2) & : \top \rightarrow A & <(1) + \vdash A \equiv (\top \rightarrow A) + \text{Eqn} > \\
(3) & : \top \rightarrow (\forall x)A & <(2) + \text{above rule} > \\
(4) & : (\forall x)A & <(3) + \vdash X \equiv (\top \rightarrow X) >
\end{align*}
\]

11. (3 marks) (Question 4, section 8.3 of book) Redo previous question for the schema

\[ (\forall x)(A \rightarrow B) \vdash (\exists x)A \rightarrow B, \text{ if } x \text{ dnof in } B \]

This is a derived rule of logic:

\[
\begin{align*}
(1) & : (\forall x)(A \rightarrow B) & <\text{hypothesis} > \\
(2) & : A \rightarrow B & <(1) + \text{spec} > \\
(3) & : (\exists x)A \rightarrow B & <(2) + \exists \text{ introduction, } x \text{ dnof in line (1) nor } B >
\end{align*}
\]