Overview

- Review: Propositions, Logical operators, Logical equivalence
- Limitations of propositional logic:
  - Refer to (constant) objects
  - Also need to say:
    - objects have certain properties
    - objects relate to one another in certain ways
- Predicate Logic is more powerful
A predicate is a proposition that is a function of one or more variables.
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Variable: $x > y$

Predicate: Relationship of the two variables

Name of the predicate: $P(x, y)$

Variables:

$x$ and $y$
Predicate (Example) - 1

Positive(x): \( x > 0 \)

What are the truth values of \( P(4) \) and \( P(-2) \)?

Solution:

\( x = 4 \)

\( P(4): \ 4 > 0 \ \square \ \text{True} \)

\( x = -2 \)

\( P(2): \ -2 > 0 \ \square \ \text{False} \)
Predicate (Example) - 5

Greater(x, y): x > y

What are the truth values of Greater(4, 1) and Greater(2, 2)?

Solution:

- x=4, y=1
  Greater(4, 1): 4 > 1 ------- True

- x=2, y=2
  Greater(2, 2): 2 > 2 ------- False
By describing the range of the variable (AKA. binding) it becomes possible to determine the truth value of the predicate.

Two popular quantifiers

- Universal: $\forall x P(x)$ - “P(x) is true for all x in the domain”

- Existential: $\exists x P(x)$ - “P(x) is true for some x in the domain”
Universal Quantifier

universal quantifier \( \forall \):

- For all...; For every...; For each...; All of...; For arbitrary...

Using universal quantifier (domain: real numbers)

- \( \forall x(x^2 \leq 0) \)

- \( (\forall x>1)(x^2>x) \) - quantifier with restricted domain
Universal Quantifier -2

∀xP(x):

☐ When true?

P(x) is true for every x in the domain

☐ When false?

P(x) is not always true when x is in the domain (find a value of x that P(x) is false)

Counterexample
Universal Quantifier
Examples

Domain: real numbers

- $P(x): x^2 \geq 0$  Is $\forall x P(x)$ true?
  - $x^2 \geq 0$ is true for all real numbers, so $\forall x P(x)$ is true

- $Q(x): x^2 > x$  Is $\forall x Q(x)$ true?
  - Find a counterexample: when $x=0$, $Q(0): 0^2 > 0$ is false, so $\forall x P(x)$ is false
Existential Quantifier

Existential quantifier \( \exists \):

- There exists...; There is...; For some...; For at least one...

Using existential quantifier (domain: real numbers)

- \( \exists x(x>1) \)
- \( \exists x(x=x+1) \)
Existential Quantifier -2

\[ \exists x P(x) : \]

- **When true?**
  
  There is an \( x \) in the domain for which \( P(x) \) is true (find a value of \( x \) that \( P(x) \) is true)

- **When false?**
  
  \( P(x) \) is false for every \( x \) in the domain
Existential Quantifier
Examples

Domain: real numbers

- \( P(x): x>1 \) Is \( \exists x P(x) \) true?
  - Find an \( x \) such that \( P(x) \) is true: when \( x=100 \), \( P(x): 100>1 \) is true, so \( \exists x P(x) \) is true

- \( Q(x): x=x+1 \) Is \( \exists x Q(x) \) true?
  - \( Q(x) \) is false for all real numbers, so \( \exists x P(x) \) is false
A tricky example

“Every CS student is smart”

\[ \forall x \ (\text{CSStudent}(x) \rightarrow \text{Smart}(x)) \]

“Some CS student is smart”

\[ \exists x \ (\text{CSStudent}(x) \land \text{Smart}(x)) \]

What is the difference of the following?

\[ \forall x \ (\text{CSStudent}(x) \land \text{Smart}(x)) \]

Why it cannot represent “Every CS student is smart”?
Scope of Quantifiers

∀, ∃ have higher precedence than operators from Propositional Logic

E.g. ∃xP(x) ∨ Q(x) is not logically equivalent to ∃x(P(x) ∨ Q(x))

<table>
<thead>
<tr>
<th>Operators</th>
<th>Precedence</th>
</tr>
</thead>
<tbody>
<tr>
<td>∀, ∃</td>
<td>0</td>
</tr>
<tr>
<td>¬</td>
<td>1</td>
</tr>
<tr>
<td>∧</td>
<td>2</td>
</tr>
<tr>
<td>∨</td>
<td>3</td>
</tr>
<tr>
<td>→</td>
<td>4</td>
</tr>
<tr>
<td>↔</td>
<td>5</td>
</tr>
</tbody>
</table>
## Logical Equivalence

<table>
<thead>
<tr>
<th></th>
<th>Logically equivalent?</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\forall x(P(x) \land Q(x))$</td>
<td>Yes</td>
</tr>
<tr>
<td>$\forall x(P(x) \lor Q(x))$</td>
<td>No</td>
</tr>
<tr>
<td>$\exists x(P(x) \land Q(x))$</td>
<td>No</td>
</tr>
<tr>
<td>$\exists x(P(x) \lor Q(x))$</td>
<td>Yes</td>
</tr>
</tbody>
</table>
Negation of Quantifiers

De Morgan’s Laws

- \( \neg \forall x P(x) = \exists x \neg P(x) \)
- \( \neg \exists x P(x) = \forall x \neg P(x) \)

**Careful:** The negation of “Every professor is good” is NOT “No professor is good”!!
Readings and Notes

Read Section 1.3

Understand the difference and relationship between propositions, predicates (functional propositions), and predicates with quantifications

Practice translating English using predicate logic

Recommended exercises: 13, 21, 23, 25, 33, 39