Solving Linear Recurrence Relations
A recursive definition of a sequence specifies

- Initial conditions
- Recurrence relation

Example:

\[ a_0 = 0 \text{ and } a_1 = 3 \]
\[ a_n = 2a_{n-1} - a_{n-2} \]
\[ a_n = 3n \]
Linear recurrences

Linear recurrence:
Each term of a sequence is a linear function of earlier terms in the sequence.

For example:

\[ a_0 = 1 \quad a_1 = 6 \quad a_2 = 10 \]
\[ a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3} \]
\[ a_3 = a_0 + 2a_1 + 3a_2 \]
\[ = 1 + 2(6) + 3(10) = 43 \]
Linear recurrences

1. Linear homogeneous recurrences
2. Linear non-homogeneous recurrences
Linear homogeneous recurrences

A linear homogeneous recurrence relation of degree $k$ with constant coefficients is a recurrence relation of the form

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k},$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

$a_n$ is expressed in terms of the previous $k$ terms of the sequence, so its degree is $k$.

This recurrence includes $k$ initial conditions.

$$a_0 = C_0 \quad a_1 = C_1 \quad \ldots \quad a_k = C_k$$
Example

Determine if the following recurrence relations are linear homogeneous recurrence relations with constant coefficients.

- $P_n = (1.11)P_{n-1}$
  - a linear homogeneous recurrence relation of degree one
- $a_n = a_{n-1} + a_{n-2}^2$
  - not linear
- $f_n = f_{n-1} + f_{n-2}$
  - a linear homogeneous recurrence relation of degree two
- $H_n = 2H_{n-1} + 1$
  - not homogeneous
- $a_n = a_{n-6}$
  - a linear homogeneous recurrence relation of degree six
- $B_n = nB_{n-1}$
  - does not have constant coefficient
Solving linear homogeneous recurrences

Proposition 1:
- Let $a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}$ be a linear homogeneous recurrence.
- Assume the sequence $a_n$ satisfies the recurrence.
- Assume the sequence $a'_n$ also satisfies the recurrence.
- So, $b_n = a_n + a'_n$ and $d_n = \alpha a_n$ are also sequences that satisfy the recurrence.
  ($\alpha$ is any constant)

Proof:
\[ b_n = a_n + a'_n \]
\[ = (c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k}) + (c_1a'_{n-1} + c_2a'_{n-2} + \ldots + c_ka'_{n-k}) \]
\[ = c_1(a_{n-1} + a'_{n-1}) + c_2(a_{n-2} + a'_{n-2}) + \ldots + c_k(a_{n-k} + a'_{n-k}) \]
\[ = c_1b_{n-1} + c_2b_{n-2} + \ldots + c_kb_{n-k} \]
So, $b_n$ is a solution of the recurrence.
Proposition 1:

- Let $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ be a linear homogeneous recurrence.
- Assume the sequence $a_n$ satisfies the recurrence.
- Assume the sequence $a'_n$ also satisfies the recurrence.
- So, $b_n = a_n + a'_n$ and $d_n = \alpha a_n$ are also sequences that satisfy the recurrence.

$(\alpha$ is any constant)

Proof:

\[
d_n = \alpha a_n \\
= \alpha (c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}) \\
= c_1 (\alpha a_{n-1}) + c_2 (\alpha a_{n-2}) + \ldots + c_k (\alpha a_{n-k}) \\
= c_1 d_{n-1} + c_2 d_{n-2} + \ldots + c_k d_{n-k}
\]

So, $d_n$ is a solution of the recurrence.
Solving linear homogeneous recurrences

It follows from the previous proposition, if we find some solutions to a linear homogeneous recurrence, then any linear combination of them will also be a solution to the linear homogeneous recurrence.
Solving linear homogeneous recurrences

Geometric sequences come up a lot when solving linear homogeneous recurrences.

So, try to find any solution of the form $a^n = r^n$ that satisfies the recurrence relation.
Solving linear homogeneous recurrences

- Recurrence relation
  \[ a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_k a_{n-k} \]

- Try to find a solution of form \( r^n \)
  \[ r^n = c_1r^{n-1} + c_2r^{n-2} + \ldots + c_k r^{n-k} \]

\[ r^n - c_1r^{n-1} - c_2r^{n-2} - \ldots - c_k r^{n-k} = 0 \]

\[ r^k - c_1r^{k-1} - c_2r^{k-2} - \ldots - c_k = 0 \] (dividing both sides by \( r^{n-k} \))

This equation is called the **characteristic equation**.
Example

Example:
The Fibonacci recurrence is
\[ F_n = F_{n-1} + F_{n-2} \]

Its characteristic equation is
\[ r^2 - r - 1 = 0 \]
Solving linear homogeneous recurrences

Proposition 2:

\[ r \text{ is a solution of } r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0 \text{ if and only if } r^n \text{ is a solution of } a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}. \]

Example:

consider the characteristic equation \( r^2 - 4r + 4 = 0. \)

\[ r^2 - 4r + 4 = (r - 2)^2 = 0 \]

So, \( r=2. \)

So, \( 2^n \) satisfies the recurrence \( F_n = 4F_{n-1} - 4F_{n-2}. \)

\[ 2^n = 4 \cdot 2^{n-1} - 4 \cdot 2^{n-2} \]

\[ 2^{n-2} (4 - 8 + 4) = 0 \]
Solving linear homogeneous recurrences

**Theorem 1:**
- Consider the characteristic equation $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$ and the recurrence $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$.
- Assume $r_1, r_2, \ldots$ and $r_m$ all satisfy the equation.
- Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be any constants.
- So, $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_m r_m^n$ satisfies the recurrence.

**Proof:**
By Proposition 2, $\forall i \ r_i^n$ satisfies the recurrence.
So, by Proposition 1, $\forall i \ \alpha_i r_i^n$ satisfies the recurrence.
Applying Proposition 1 again, the sequence $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_m r_m^n$ satisfies the recurrence.
Example

What is the solution of the recurrence relation

\[ a_n = a_{n-1} + 2a_{n-2} \]

with \( a_0 = 2 \) and \( a_1 = 7 \)?

Solution:

- Since it is linear homogeneous recurrence, first find its characteristic equation
  \[ r^2 - r - 2 = 0 \]
  \[ (r+1)(r-2) = 0 \]
  \[ r_1 = 2 \] and \( r_2 = -1 \)
- So, by theorem \( a_n = \alpha_1 2^n + \alpha_2 (-1)^n \) is a solution.
- Now we should find \( \alpha_1 \) and \( \alpha_2 \) using initial conditions.
  \[ a_0 = \alpha_1 + \alpha_2 = 2 \]
  \[ a_1 = \alpha_1 2 + \alpha_2 (-1) = 7 \]
- So, \( \alpha_1 = 3 \) and \( \alpha_2 = -1 \).
- \( a_n = 3 \cdot 2^n - (-1)^n \) is a solution.
Example

What is the solution of the recurrence relation

\[ f_n = f_{n-1} + f_{n-2} \]

with \( f_0 = 0 \) and \( f_1 = 1 \)?

Solution:

- Since it is linear homogeneous recurrence, first find its characteristic equation

  \[ r^2 - r - 1 = 0 \]

  \[ r_1 = \frac{1 + \sqrt{5}}{2} \] and \( r_2 = \frac{1 - \sqrt{5}}{2} \)

- So, by theorem \( f_n = \alpha_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + \alpha_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n \) is a solution.

- Now we should find \( \alpha_1 \) and \( \alpha_2 \) using initial conditions.

  \[ f_0 = \alpha_1 + \alpha_2 = 0 \]

  \[ f_1 = \alpha_1 \left( \frac{1 + \sqrt{5}}{2} \right) + \alpha_2 \left( \frac{1 - \sqrt{5}}{2} \right) = 1 \]

- So, \( \alpha_1 = \frac{1}{\sqrt{5}} \) and \( \alpha_2 = -\frac{1}{\sqrt{5}} \).

- \( a_n = \frac{1}{\sqrt{5}} \cdot \left( \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n \right) \) is a solution.
Example

What is the solution of the recurrence relation

\[ a_n = -a_{n-1} + 4a_{n-2} + 4a_{n-3} \]

with \( a_0 = 8 \), \( a_1 = 6 \) and \( a_2 = 26 \)?

Solution:

- Since it is linear homogeneous recurrence, first find its characteristic equation

\[
(r+1)(r+2)(r-2) = 0 \quad r_1 = -1, \ r_2 = -2 \text{ and } r_3 = 2
\]

- So, by theorem \( a_n = \alpha_1(-1)^n + \alpha_2(-2)^n + \alpha_32^n \) is a solution.

- Now we should find \( \alpha_1 \), \( \alpha_2 \) and \( \alpha_3 \) using initial conditions.

\[
a_0 = \alpha_1 + \alpha_2 + \alpha_3 = 8
\]
\[
a_1 = -\alpha_1 - 2\alpha_2 + 2\alpha_3 = 6
\]
\[
a_2 = \alpha_1 + 4\alpha_2 + 4\alpha_3 = 26
\]

- So, \( \alpha_1 = 2 \), \( \alpha_2 = 1 \) and \( \alpha_3 = 5 \).

- \( a_n = 2 \cdot (-1)^n + (-2)^n + 5 \cdot 2^n \) is a solution.
Solving linear homogeneous recurrences

If the characteristic equation has k distinct solutions $r_1, r_2, \ldots, r_k$, it can be written as

$$(r - r_1)(r - r_2)\ldots(r - r_k) = 0.$$ 

If, after factoring, the equation has $m+1$ factors of $(r - r_1)$, for example, $r_1$ is called a solution of the characteristic equation with multiplicity $m+1$.

When this happens, not only $r_1^n$ is a solution, but also $nr_1^n, n^2r_1^n, \ldots$ and $n^mr_1^n$ are solutions of the recurrence.
Proposition 3:

- Assume $r_0$ is a solution of the characteristic equation with multiplicity at least $m+1$.
- So, $n^m r_0^n$ is a solution to the recurrence.
Solving linear homogeneous recurrences

When a characteristic equation has fewer than $k$ distinct solutions:

- We obtain sequences of the form described in Proposition 3.
- By Proposition 1, we know any combination of these solutions is also a solution to the recurrence.
- We can find those that satisfies the initial conditions.
Solving linear homogeneous recurrences

Theorem 2:

- Consider the characteristic equation \( r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0 \) and the recurrence \( a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} \).

- Assume the characteristic equation has \( t \leq k \) distinct solutions.

- Let \( \forall i \ (1 \leq i \leq t) \ r_i \) with multiplicity \( m_i \) be a solution of the equation.

- Let \( \forall i,j \ (1 \leq i \leq t \text{ and } 0 \leq j \leq m_i-1) \ \alpha_{ij} \) be a constant.

- So, \( a_n = (\alpha_{10} + \alpha_{11} n + \ldots + \alpha_{1,m_1-1} n^{m_1-1}) r_1^n + (\alpha_{20} + \alpha_{21} n + \ldots + \alpha_{2,m_2-1} n^{m_2-1}) r_2^n + \ldots + (\alpha_{t0} + \alpha_{t1} n + \ldots + \alpha_{t,m_t-1} n^{m_t-1}) r_t^n \) satisfies the recurrence.
Example

What is the solution of the recurrence relation
\[ a_n = 6a_{n-1} - 9a_{n-2} \]
with \( a_0=1 \) and \( a_1=6 \)?

Solution:

- First find its characteristic equation
  \[ r^2 - 6r + 9 = 0 \]
  \[ (r - 3)^2 = 0 \quad r_1 = 3 \] (Its multiplicity is 2.)

- So, by theorem \( a_n = (\alpha_{10} + \alpha_{11}n)(3)^n \) is a solution.

- Now we should find constants using initial conditions.
  \[ a_0 = \alpha_{10} = 1 \]
  \[ a_1 = 3 \alpha_{10} + 3\alpha_{11} = 6 \]
  \[ \alpha_{11} = 1 \text{ and } \alpha_{10} = 1. \]
  \[ a_n = 3^n + n3^n \text{ is a solution.} \]
Example

What is the solution of the recurrence relation
\[ a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3} \]
with \( a_0=1, \ a_1=-2 \) and \( a_2=-1 \)?

Solution:

- Find its characteristic equation
  \[ r^3 + 3r^2 + 3r + 1 = 0 \]
  \[ (r + 1)^3 = 0 \quad r_1 = -1 \quad (\text{Its multiplicity is 3.}) \]

- So, by theorem \( a_n = (\alpha_{10} + \alpha_{11}n + \alpha_{12}n^2)(-1)^n \) is a solution.

- Now we should find constants using initial conditions.
  \[ a_0 = \alpha_{10} = 1 \]
  \[ a_1 = -\alpha_{10} - \alpha_{11} - \alpha_{12} = -2 \]
  \[ a_2 = \alpha_{10} + 2\alpha_{11} + 4\alpha_{12} = -1 \]

- So, \( \alpha_{10} = 1, \ \alpha_{11} = 3 \) and \( \alpha_{12} = -2 \).

- \( a_n = (1 + 3n - 2n^2)(-1)^n \) is a solution.
Example

What is the solution of the recurrence relation
\[ a_n = 8a_{n-2} - 16a_{n-4}, \text{ for } n \geq 4, \]
with \( a_0 = 1, \ a_1 = 4, \ a_2 = 28 \) and \( a_3 = 32 \)?

Solution:

- Find its characteristic equation
  \[ r^4 - 8r^2 + 16 = 0 \]
  \[ (r^2 - 4)^2 = (r-2)^2 (r+2)^2 = 0 \]
  \[ r_1 = 2 \quad r_2 = -2 \quad \text{(Their multiplicities are 2.)} \]

- So, by theorem \( a_n = (a_{10} + a_{11}n)(2)^n + (a_{20} + a_{21}n)(-2)^n \) is a solution.

- Now we should find constants using initial conditions.
  \[ a_0 = \alpha_{10} + \alpha_{20} = 1 \]
  \[ a_1 = 2\alpha_{10} + 2\alpha_{11} - 2\alpha_{20} - 2\alpha_{21} = 4 \]
  \[ a_2 = 4\alpha_{10} + 8\alpha_{11} + 4\alpha_{20} + 8\alpha_{21} = 28 \]
  \[ a_3 = 8\alpha_{10} + 24\alpha_{11} - 8\alpha_{20} - 24\alpha_{21} = 32 \]

- So, \( \alpha_{10} = 1, \ a_{11} = 2, \ a_{20} = 0 \) and \( a_{21} = 1 \).

- \( a_n = (1 + 2n) \ 2^n + n (-2)^n \) is a solution.
A linear non-homogeneous recurrence relation with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + f(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $f(n)$ is a function depending only on $n$.

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k},$$

is called the associated homogeneous recurrence relation.

This recurrence includes $k$ initial conditions.

$$a_0 = C_0 \quad a_1 = C_1 \quad \ldots \quad a_k = C_k$$
Example

The following recurrence relations are linear non-homogeneous recurrence relations.

- \( a_n = a_{n-1} + 2^n \)
- \( a_n = a_{n-1} + a_{n-2} + n^2 + n + 1 \)
- \( a_n = a_{n-1} + a_{n-4} + n! \)
- \( a_n = a_{n-6} + n2^n \)
Linear non-homogeneous recurrences

Proposition 4:

- Let \( a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + f(n) \) be a linear non-homogeneous recurrence.
- Assume the sequence \( b_n \) satisfies the recurrence.
- Another sequence \( a_n \) satisfies the non-homogeneous recurrence if and only if \( h_n = a_n - b_n \) is also a sequence that satisfies the associated homogeneous recurrence.
Linear non-homogeneous recurrences

Proof:
Part1: if $h_n$ satisfies the associated homogeneous recurrence then $a_n$ is satisfies the non-homogeneous recurrence.

- $b_n = c_1 b_{n-1} + c_2 b_{n-2} + \ldots + c_k b_{n-k} + f(n)$
- $h_n = c_1 h_{n-1} + c_2 h_{n-2} + \ldots + c_k h_{n-k}$

$$b_n + h_n = c_1 (b_{n-1} + h_{n-1}) + c_2 (b_{n-2} + h_{n-2}) + \ldots + c_k (b_{n-k} + h_{n-k}) + f(n)$$

Since $a_n = b_n + h_n$, $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + f(n)$.
So, $a_n$ is a solution of the non-homogeneous recurrence.
Linear non-homogeneous recurrences

Proof:
Part 2: if $a_n$ satisfies the non-homogeneous recurrence then $h_n$ is satisfies the associated homogeneous recurrence.

- $b_n = c_1 b_{n-1} + c_2 b_{n-2} + \ldots + c_k b_{n-k} + f(n)$
- $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} + f(n)$

$$a_n - b_n = c_1 (a_{n-1} - b_{n-1}) + c_2 (a_{n-2} - b_{n-2}) + \ldots + c_k (a_{n-k} - b_{n-k})$$

Since $h_n = a_n - b_n$, $h_n = c_1 h_{n-1} + c_2 h_{n-2} + \ldots + c_k h_{n-k}$

So, $h_n$ is a solution of the associated homogeneous recurrence.
Linear non-homogeneous recurrences

Proposition 4:
- Let \( a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k} + f(n) \) be a linear non-homogeneous recurrence.
- Assume the sequence \( b_n \) satisfies the recurrence.
- Another sequence \( a_n \) satisfies the non-homogeneous recurrence if and only if \( h_n = a_n - b_n \) is also a sequence that satisfies the associated homogeneous recurrence.

- We already know how to find \( h_n \).
- For many common \( f(n) \), a solution \( b_n \) to the non-homogeneous recurrence is similar to \( f(n) \).
- Then you should find solution \( a_n = b_n + h_n \) to the non-homogeneous recurrence that satisfies both recurrence and initial conditions.
Example

What is the solution of the recurrence relation
\[ a_n = a_{n-1} + a_{n-2} + 3n + 1 \]
for \( n \geq 2 \), with \( a_0 = 2 \) and \( a_1 = 3 \)?

Solution:

- Since it is linear non-homogeneous recurrence, \( b_n \) is similar to \( f(n) \)
  
  Guess: \( b_n = cn + d \)
  
  \[ b_n = b_{n-1} + b_{n-2} + 3n + 1 \]
  
  \[ cn + d = c(n-1) + d + c(n-2) + d + 3n + 1 \]
  
  \[ cn + d = cn - c + d + cn - 2c + d + 3n + 1 \]
  
  \[ 0 = (3+c)n + (d-3c+1) \]
  
  \[ c = -3 \quad d = -10 \]

- So, \( b_n = -3n - 10 \).
  
  (\( b_n \) only satisfies the recurrence, it does not satisfy the initial conditions.)
Example

What is the solution of the recurrence relation
\[ a_n = a_{n-1} + a_{n-2} + 3n + 1 \text{ for } n \geq 2, \]
with \( a_0 = 2 \) and \( a_1 = 3 \)?

Solution:

- We are looking for \( a_n \) that satisfies both recurrence and initial conditions.
- \( a_n = b_n + h_n \) where \( h_n \) is a solution for the associated homogeneous recurrence: \( h_n = h_{n-1} + h_{n-2} \)
- By previous example, we know \( h_n = \alpha_1((1+\sqrt{5})/2)^n + \alpha_2((1-\sqrt{5})/2)^n \).
- \( a_n = b_n + h_n \)
  \[ = -3n - 10 + \alpha_1((1+\sqrt{5})/2)^n + \alpha_2((1-\sqrt{5})/2)^n \]
- Now we should find constants using initial conditions.
  \[ a_0 = -10 + \alpha_1 + \alpha_2 = 2 \]
  \[ a_1 = -13 + \alpha_1 (1+\sqrt{5})/2 + \alpha_2 (1-\sqrt{5})/2 = 3 \]
  \[ \alpha_1 = 6 + 2 \sqrt{5} \quad \alpha_2 = 6 - 2 \sqrt{5} \]
  
  So, \( a_n = -3n - 10 + (6 + 2 \sqrt{5})((1+\sqrt{5})/2)^n + (6 - 2 \sqrt{5})((1-\sqrt{5})/2)^n \).
What is the solution of the recurrence relation
\[ a_n = 2a_{n-1} - a_{n-2} + 2^n \] for \( n \geq 2 \),
with \( a_0=1 \) and \( a_1=2 \)?

Solution:

- Since it is linear non-homogeneous recurrence, \( b_n \) is similar to \( f(n) \)

  Guess: \( b_n = c2^n + d \)

\[
b_n = 2b_{n-1} - b_{n-2} + 2^n
\]

\[
c2^n + d = 2(c2^{n-1} + d) - (c2^{n-2} + d) + 2^n
\]

\[
c2^n + d = c2^n + 2d - c2^{n-2} - d + 2^n
\]

\[
0 = (-4c + 4c - c + 4)2^{n-2} + (-d + 2d -d)
\]

\[
c = 4 \quad d=0
\]

- So, \( b_n = 4 \cdot 2^n \).

  (\( b_n \) only satisfies the recurrence, it does not satisfy the initial conditions.)
Example

What is the solution of the recurrence relation

\[ a_n = 2a_{n-1} - a_{n-2} + 2^n \text{ for } n \geq 2, \]

with \( a_0 = 1 \) and \( a_1 = 2 \) ?

Solution:

- We are looking for \( a_n \) that satisfies both recurrence and initial conditions.
- \( a_n = b_n + h_n \) where \( h_n \) is a solution for the associated homogeneous recurrence: \( h_n = 2h_{n-1} - h_{n-2} \).
  - Find its characteristic equation
    \[ r^2 - 2r + 1 = 0 \]
    \[ (r - 1)^2 = 0 \]
    \[ r_1 = 1 \] (Its multiplicity is 2.)
- So, by theorem \( h_n = (\alpha_1 + \alpha_2 n)(1)^n = \alpha_1 + \alpha_2 n \) is a solution.
Example

What is the solution of the recurrence relation

\[ a_n = 2a_{n-1} - a_{n-2} + 2^n \text{ for } n \geq 2, \]

with \( a_0 = 1 \) and \( a_1 = 2 \)?

Solution:

- \( a_n = b_n + h_n \)
- \( a_n = 4 \cdot 2^n + \alpha_1 + \alpha_2 n \) is a solution.
- Now we should find constants using initial conditions.

\[
\begin{align*}
  a_0 &= 4 + \alpha_1 = 1 \\
  a_1 &= 8 - \alpha_1 + \alpha_2 = 2 \\
  \alpha_1 &= -3 \\
  \alpha_2 &= -3 \\
\end{align*}
\]

So, \( a_n = 4 \cdot 2^n - 3n - 3 \).
Recommended exercises

1, 3, 15, 17, 19, 21, 23, 25, 31, 35

Eric Ruppert’s Notes about Solving Recurrences

(http://www.cse.yorku.ca/course_archive/2007-08/F/1019/A/recurrence.pdf)