Mathematical induction

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Mathematical induction

- Mathematical induction is an extremely important proof technique.

- Mathematical induction can be used to prove
  - results about complexity of algorithms
  - correctness of certain types of computer programs
  - theorem about graphs and trees
  - …

- Mathematical induction can be used only to prove results obtained in some other ways.

Mathematical induction

Assume P(n) is a propositional function.

Principle of mathematical induction:

To prove that P(n) is true for all positive integers n we complete two steps

1. Basis step:

Verify P(1) is true.

2. Inductive step:

Show $P(k) \rightarrow P(k+1)$ is true for all positive integers k.

Mathematical induction

Basis step: $P(1)$

Inductive step: $\forall k\,(P(k) \rightarrow P(k+1))$

Result: $\forall n\, P(n)$ domain: positive integers

1. $P(1)$
2. $\forall k\,(P(k) \rightarrow P(k+1))$
3. $P(1) \rightarrow P(2)$
4. $P(2)$ by Modus ponens
5. $P(2) \rightarrow P(3)$
6. $P(3)$ by Modus ponens

   …

Mathematical induction

$$[P(1) \wedge \forall k\, (P(k) \rightarrow P(k+1))) \rightarrow \forall n\, P(n)]$$

How to show P(1) is true?

- P(1): n is replaced by 1 in P(n)
- Then, show P(1) is true.

How to show $\forall k\, (P(k) \rightarrow P(k+1))$?

- Direct proof can be used
- Assume P(k) is true for some arbitrary k.
- Then, show P(k+1) is true.

Example

Show that 1+2+…+n = n(n+1)/2, where n is a positive integer.

Proof by induction:

- First define P(n)

  P(n) is 1+2+…+n= n(n+1)/2
- Basis step: (Show P(1) is true.)

  1 = 1(2)/2

  So, P(1) is true.

Example

Show that 1+2+…+n = n(n+1)/2, where n is a positive integer.

Proof by induction:

- Inductive step: (Show $\forall k\ (P(k) \rightarrow P(k+1))$ is true.)
  - Assume P(k) is true.

    1+2+…+k = k(k+1)/2
  - Show P(k+1) is true.

    P(k+1) is 1+2+…+k+(k+1)=(k+1)(k+2)/2

  1+2+…+k+(k+1) = k(k+1)/2 + (k+1)

  $= [k(k+1) + 2(k+1)] / 2 = [k^2 + k + 2k +2] / 2$

  $= [k^2 + 3k +2]/2 = (k+1)(k+2)/2$

We showed that P(k+1) is true under assumption that P(k) is true.

So, by mathematical induction 1+2+…+n = n(n+1)/2.

Example

Show that $1+3+5\ldots+(2n-1) = n^2$, where n is a positive integer.

Proof by induction:

- First define P(n)

  P(n) is $1+3+5\ldots+(2n-1) = n^2$

- Basis step: (Show P(1) is true.)

  $2-1 = 1^2$

  So, P(1) is true.

Example

Show that $1+3+5\ldots+(2n-1) = n^2$, where n is a positive integer.

Proof by induction:

- Inductive step: (Show $\forall k\ (P(k) \rightarrow P(k+1))$ is true.)
  - Assume P(k) is true.

    $1+3+5\ldots+(2k-1) = k^2$
  - Show P(k+1) is true.

    P(k+1) is $1+3+5\ldots+(2k-1)+(2(k+1)-1) = (k+1)^2$

    $1+3+5\ldots+(2k-1)+(2(k+1)-1) = k^2 + (2(k+1)-1)$

    $= k^2 + 2k + 1$

    $= (k+1)^2$

  We showed that P(k+1) is true under assumption that P(k) is true.

So, by mathematical induction $1+3+5\ldots+(2n-1) = n^2$.

Mathematical induction

Sometimes we need to show that $P(n)$ is true for $n = b, b+1, b+2, \dots$, where b is an integer other than 1.

Mathematical induction:

- **Basis step:**
  - Show $P(b)$ is true.
- **Inductive step:**
  - Show $\forall k\ (P(k) \rightarrow P(k+1))$ is true.

Example

Use mathematical induction to show that
$1+2+2^2+\ldots+2^n = 2^{n+1} - 1$ for all nonnegative integers n.

Proof by induction:

- First define P(n)

  P(n) is $2^0+2^1+2^2+\ldots+2^n = 2^{n+1} - 1$
- Basis step: (Show P(0) is true.)

  $2^0 = 2^1 - 1$

  So, P(0) is true.

Example

Use mathematical induction to show that $1+2+2^2+\ldots+2^n = 2^{n+1} - 1$ for all nonnegative integers n.

Proof by induction:

- Inductive step: (Show $\forall k\ (P(k) \rightarrow P(k+1))$ is true.)
  - Assume P(k) is true.

    $1+2+2^2+\ldots+2^k = 2^{k+1} - 1$
  - Show P(k+1) is true.

    P(k+1) is $1+2+2^2+\ldots+2^{k+1} = 2^{k+2} - 1$

$$1+2+2^2+\ldots+2^k+2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$$
$$= 2 \cdot 2^{k+1} - 1$$
$$= 2^{k+2} - 1$$

We showed that P(k+1) is true under assumption that P(k) is true.

So, by mathematical induction $1+2+2^2+\ldots+2^n = 2^{n+1} - 1$.

Example

Use mathematical induction to prove the formula for the sum of a finite number of terms of a geometric progression.

$$\sum_{k=0}^{n} ar^k = a+ar+ar^2+\ldots+ar^n = (ar^{n+1} - a) / (r-1) \text{ when } r \neq 1$$

Proof by induction:

- First define P(n)

  P(n) is $a+ar+ar^2+\ldots+ar^n = (ar^{n+1} - a) / (r-1)$.

- Basis step: (Show P(0) is true.)

  $ar^0 = (ar - a)/(r-1) = a$

  So, P(0) is true.

Example

Use mathematical induction to prove the formula for the sum of a finite number of terms of a geometric progression.

$$\sum_{k=0}^{n} ar^k = a+ar+ar^2+\ldots+ar^n = (ar^{n+1} - a) / (r-1) \text{ when } r \neq 1$$

Proof by induction:

- Inductive step: (Show $\forall k\,(P(k) \rightarrow P(k+1))$ is true.)
  - Assume P(k) is true.

    $a+ar+ar^2+\ldots+ar^k = (ar^{k+1} - a) / (r-1)$
  - Show P(k+1) is true.

    P(k+1) is $a+ar+ar^2+\ldots+ar^{k+1} = (ar^{k+2} - a) / (r-1)$

Example

Use mathematical induction to prove the formula for the sum of a finite number of terms of a geometric progression.

$$\sum_{k=0}^{n} ar^k = a+ar+ar^2+\ldots+ar^n = (ar^{n+1} - a) / (r-1) \text{ when } r \neq 1$$

Proof by induction:

$$a+ar+ar^2+\ldots+ar^k+ar^{k+1} = (ar^{k+1} - a) / (r-1) + ar^{k+1}$$

$$= (ar^{k+1} - a) / (r-1) + ar^{k+1} (r-1) / (r-1)$$

$$= (ar^{k+1} - a + ar^{k+2} - ar^{k+1}) / (r-1)$$

$$= (ar^{k+2} - a) / (r-1)$$

We showed that P(k+1) is true under assumption that P(k) is true.

So, by mathematical induction $a+ar+ar^2+\ldots+ar^n = (ar^{n+1} - a) / (r-1)$.

Example

Use mathematical induction to prove that $2^n < n!$ for every positive integer n with $n \geq 4$.

Proof by induction:

- First define P(n)

  P(n) is $2^n < n!$.

- Basis step: (Show P(4) is true.)

  $2^4 < 1 . 2 . 3 . 4$

  $16 < 24$

  So, P(4) is true.

Example

Use mathematical induction to prove that $2^n<n!$ for every positive integer n with $n \geq 4$.

Proof by induction:

- Inductive step: (Show $\forall k\ (P(k) \rightarrow P(k+1))$ is true.)
  - Assume P(k) is true.

    $2^k < k!$
  - Show P(k+1) is true.

    P(k+1) is $2^{(k+1)} < (k+1)!$

    $2 . 2^k < 2 . k!$

    $2^{(k+1)} < 2 . k!$

    $< (k+1) . k! = (k+1)!$

  We showed that P(k+1) is true under assumption that P(k) is true.

So, by mathematical induction $2^n<n!$.

Example

Harmonic numbers H_j, j=1,2,3,… are defined by $H_j = 1+1/2 + 1/3 + \ldots + 1/j$.

Use mathematical induction to show that $H_{2^n} \geq 1 + n/2$, whenever n is a nonnegative integer.

Proof by induction:

- First define P(n)

  P(n) is $H_{2^n} \geq 1+n/2$.

- Basis step: (Show P(0) is true.)

  $H_{2^0} \geq 1 + 0/2$

  $1 \geq 1$

  So, P(0) is true.

Example

Use mathematical induction to show that $H_{2^n} \geq 1 + n/2$, whenever n is a nonnegative integer.

Proof by induction:

- Inductive step: (Show $\forall k\ (P(k) \rightarrow P(k+1))$ is true.)
  - Assume P(k) is true.

    $H_{2^k} = 1+1/2+1/3+\ldots+1/2^k \geq 1 + k/2$
  - Show P(k+1) is true.

    P(k+1) is $1+1/2+1/3+\ldots+1/2^k + 1/(2^k+1) + \ldots + 1/2^{k+1} \geq 1+(k+1)/2$

$$1+1/2+1/3+\ldots+1/2^k + 1/(2^k+1) + \ldots + 1/2^{k+1} \geq 1+k/2+ 1/(2^k+1) + \ldots + 1/2^{k+1}$$

$$\geq (1 + k/2) + 2^k \cdot 1/2^{k+1}$$

$$\geq (1 + k/2) + 1/2 = 1 + (k+1)/2$$

We showed that P(k+1) is true under assumption that P(k) is true.

So, by mathematical induction $H_{2^n} \geq 1 + n/2$.

Example

Use mathematical induction to prove that n^3-n is divisible by 3 whenever n is a positive integer.

Proof by induction:

- First define P(n)

  P(n) is "n^3-n is divisible by 3".

- Basis step: (Show P(1) is true.)

  1^3-1 = 0 is divisible by 3.

  So, P(1) is true.

Example

Use mathematical induction to prove that n^3-n is divisible by 3 whenever n is a positive integer.

Proof by induction:

- Inductive step: (Show $\forall k\ (P(k) \rightarrow P(k+1))$ is true.)
  - Assume P(k) is true. (k^3-k is divisible by 3)
  - Show P(k+1) is true. (P(k+1) is $(k+1)^3 - (k+1)$ is divisible by 3.)

$$(k+1)^3 - (k+1) = (k^3 + 3k^2 + 3k + 1) - (k+1)$$

$$= (k^3 - k) + 3(k^2 + k)$$

By inductive hypothesis $(k^3 - k)$ is divisible by 3 and $3(k^2 + k)$ is divisible by 3 because it is 3 times an integer, so P(k+1) is divisible by 3

We showed that P(k+1) is true under assumption that P(k) is true.

So, by mathematical induction n^3-n is divisible by 3.

Example

Let S be a set with n elements, where n is nonnegative integer. Use mathematical induction to show that S has 2^n subsets.

Proof by induction:

- First define P(n)

  P(n) is “A set with n elements has 2^n subsets“.

- Basis step: (Show P(0) is true.)

  The empty set has $2^0=1$ subset, namely, itself.

  So, P(0) is true.

Example

Let S be a set with n elements, where n is nonnegative integer. Use mathematical induction to show that S has 2^n subsets.

Proof by induction:

- Inductive step: (Show ∀k (P(k) →P(k+1)) is true.)
  - Assume P(k) is true. (set S with k elements has 2^k subsets)
  - Show P(k+1) is true. (set T (=S∪{a}) has 2^{k+1} subsets.)

    For each subset X of S there are exactly two subsets of T, namely, X and X∪{a}.

    Since S has 2^k subsets, T has $2 \cdot 2^k = 2^{k+1}$ subsets.

We showed that P(k+1) is true under assumption that P(k) is true.

So, by mathematical induction, any set with n elements, has 2^n subsets.

Example

Use mathematical induction to prove the following generalization of one of De Morgan's laws:

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$$

when n≥2.

Proof by induction:

- First define P(n)

  P(n) is $\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j}$.

- Basis step: (Show P(2) is true.)

  $\overline{A_1 \cap A_2} = \overline{A_1} \cup \overline{A_2}$

  By De Morgan's law, P(2) is true.

Example

Use mathematical induction to prove

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j} \qquad \text{when } n \geq 2.$$

Proof by induction:

- Inductive step: (Show ∀k (P(k) →P(k+1)) is true.)
  - Assume P(k) is true.

$$\overline{\bigcap_{j=1}^{k} A_j} = \bigcup_{j=1}^{k} \overline{A_j}$$

  - Show P(k+1) is true.

$$\overline{\bigcap_{j=1}^{k+1} A_j} = \bigcup_{j=1}^{k+1} \overline{A_j}$$

Example

Use mathematical induction to prove

$$\overline{\bigcap_{j=1}^{n} A_j} = \bigcup_{j=1}^{n} \overline{A_j} \qquad \text{when } n \geq 2.$$

Proof by induction:

$$\overline{\bigcap_{j=1}^{k+1} A_j} = \overline{\left(\bigcap_{j=1}^{k} A_j\right) \cap A_{k+1}}$$

$$= \overline{\left(\bigcap_{j=1}^{k} A_j\right)} \cup \overline{A_{k+1}} \quad \text{(by De Morgan's law)}$$

$$= \left(\bigcup_{j=1}^{k} \overline{A_j}\right) \cup \overline{A_{k+1}} \quad \text{(by inductive hypothesis)}$$

$$= \bigcup_{j=1}^{k+1} \overline{A_j}$$

We showed that P(k+1) is true under assumption that P(k) is true.

Example

- An odd number of people stand in a yard at mutually distinct distances.
- At the same time each person throws a pie at their nearest neighbor, hitting this person.

  Use mathematical induction to show that there is at least one person who is not hit by a pie.

Proof by induction:

- First define P(n)

  P(n) is “there is one survivor whenever 2n+1 people stand in a yard at distinct mutual distances and each person throws a pie at their nearest neighbor”.

Example

Proof by induction:

- Basis step: (Show P(1) is true.)
  - There are 2(1)+1=3 people (A,B and C) in the pie fight.
  - Assume the closest pair is A and B.
  - Since the distances between pairs of people are different, the distance between A and C and the distance between B and C are greater than the distance between A and B.
  - So, A and B throw a pie at each other, while C throws a pie at either A or B, whichever is closer.
  - So, C is not hit by a pie and P(1) is true.

Example

Proof by induction:

- Inductive step: (Show $\forall k\ (P(k) \rightarrow P(k+1))$ is true.)
  - Assume P(k) is true.

    There is at least one survivor whenever 2k+1 people stand in a yard at distinct mutual distances and each throws a pie at their nearest neighbor.
  - Show P(k+1) is true.
    - Assume there are 2(k+1)+1=2k+3 people in a yard with distinct distance between pairs of people.
    - Let A and B be the closest pair of people among 2k+3 people.
    - So, A and B throw pies at each other.

Example

Proof by induction:

- **Case 1:**
  - If someone else throws a pie at either A or B.
    - So, three pies are thrown at A and B.
    - So, at most (2k+3 - 3) = 2k pies are thrown at the remaining 2k+1 people.
    - This guarantees that at least one person is a survivor.
- **Case 2:**
  - If no one else throws a pie at either A or B.
  - Besides A and B, there are 2k+1 people.
  - Since the distances between pairs of people are all different, by inductive hypothesis, there is at least one survivor when 2k+1 people throws pie at each other.

So, by mathematical induction, P(n) is true.

Recommended exercises

3,6,7,11,13,16,21,27,29,33,35,39,41,43,45,49,59,61,70