Nested Quantifiers
Nested quantifiers

Two quantifiers are nested if one is within the scope of the other.

Example:

\[ \forall x \exists y (x + y = 0) \]

\[ \forall x \ Q(x) \]

\[ Q(x) \text{ is } \exists y \ P(x,y) \]

\[ P(x,y) \text{ is } (x + y = 0) \]
Nested quantifiers (example)

Translate the following statement into English.

∀x ∀y (x + y = y + x)

Domain: real numbers

Solution:

For all real numbers x and y, x + y = y + x.
Nested quantifiers (example)

Translate the following statement into English.

\( \forall x \ \exists y \ (x = -y) \)

Domain: real numbers

Solution:

For every real number \( x \), there is a real number \( y \) such that \( x = -y \).
Nested quantifiers (example)

Translate the following statement into English.
\( \forall x \ \forall y \ ((x > 0) \land (y < 0) \rightarrow (xy < 0)) \)

Domain: real numbers

Solution:
For every real numbers \( x \) and \( y \), if \( x \) is positive and \( y \) is negative then \( xy \) is negative.

The product of a positive real number and a negative real number is always a negative real number.
The order of quantifiers (example)

Assume \( P(x,y) \) is \((xy = yx)\).
Translate the following statement into English.
\( \forall x \ \forall y \ P(x,y) \) domain: real numbers

Solution:
For all real numbers \( x \), for all real numbers \( y \),
\( xy = yx \).

For every pair of real numbers \( x, y \), \( xy = yx \).
The order of quantifiers

The order of nested *universal* quantifiers in a statement *without* other quantifiers can be changed without changing the meaning of the quantified statement.
The order of quantifiers (example)

Assume $P(x,y)$ is $(xy = 6)$.
Translate the following statement into English.
$\exists x \\exists y \ P(x,y)$  

Solution:

There is an integer $x$ for which there is an integer $y$ that $xy = 6$.

There is a pair of integers $x, y$ for which $xy = 6$. 
The order of quantifiers

The order of nested *existential* quantifiers in a statement *without* other quantifiers can be changed without changing the meaning of the quantified statement.
The order of quantifiers (example)

Assume $P(x,y)$ is $(x + y = 10)$.

$\forall x \ \exists y \ P(x,y)$ \hspace{1cm} \text{domain: real numbers}
For all real numbers $x$ there is a real number $y$ such that $x + y = 10$.

True \hspace{1cm} (y = 10 - x)

$\exists y \ \forall x \ P(x,y)$ \hspace{1cm} \text{domain: real numbers}
There is a real number $y$ such that for all real numbers $x$, $x + y = 10$.

False
So, $\forall x \ \exists y \ P(x,y)$ and $\exists y \ \forall x \ P(x,y)$ are not logically equivalent.
The order of quantifiers

Assume $P(x,y,z)$ is $(x + y = z)$.

$\forall x \ \forall y \ \exists z \ P(x,y,z)$ \hspace{1cm} domain: real numbers

For all real numbers $x$ and $y$ there is a real number $z$ such that $x + y = z$.

True

$\exists z \ \forall x \ \forall y \ P(x,y,z)$ \hspace{1cm} domain: real numbers

There is a real number $z$ such that for all real numbers $x$ and $y$ $x + y = z$.

False

So, $\forall x \ \forall y \ \exists z \ P(x,y,z)$ and $\exists z \ \forall x \ \forall y \ P(x,y,z)$ are not logically equivalent.
The order of quantifiers

The order of nested existential and universal quantifiers in a statement is important.
Quantification of two variable

\( \forall x \forall y \ P(x,y) \)
- When true?
  \( P(x,y) \) is true for every pair \( x,y \).
- When false?
  There is a pair \( x, y \) for which \( P(x,y) \) is false.

\( \exists x \forall y \ P(x,y) \)
- When true?
  For every \( x \) there is a \( y \) for which \( P(x,y) \) is true.
- When false?
  There is an \( x \) such that \( P(x,y) \) is false for every \( y \).
Quantification of two variable

- **∃x ∀y P(x,y)**
  - When true?
    There is an x for which P(x,y) is true for every y.
  - When false?
    For every x there is a y for which P(x,y) is false.

- **∃x ∃y P(x,y)**
  - When true?
    There is a pair x, y for which P(x,y) is true.
  - When false?
    P(x,y) is false for every pair x, y.
Nested quantifiers (example)

Translate the following statement into a logical expression.

“The sum of two positive integers is always positive.”

Solution:
- Rewrite it in English that quantifiers and a domain are shown

  “For every pair of integers, if both integers are positive, then the sum of them is positive.”
Nested quantifiers (example)

Translate the following statement into a logical expression.
“The sum of two positive integers is always positive.”

Solution:
- Introduce variables
  “For every pair of integers, if both integers are positive, then the sum of them is positive.”

  “For all integers x, y, if x and y are positive, then x+y is positive.”
Nested quantifiers (example)

Translate the following statement into a logical expression. “The sum of two positive integers is always positive.”

Solution:

Translate it to a logical expression

“For all integers x, y, if x and y are positive, then x+y is positive.”

∀x ∀y ((x > 0) ∧ (y > 0) → (x + y > 0)) domain: integers

∀x ∀y (x + y > 0) domain: positive integers
Nested quantifiers (example)

Translate the following statement into a logical expression.
“Every real number except zero has a multiplicative inverse.”

A multiplicative inverse of a real number \( x \) is a real number \( y \) such that \( xy = 1 \).

Solution:

- Rewrite it in English that **quantifiers** and a **domain** are shown

  “For **every real number** except zero, there is a multiplicative inverse.”
Nested quantifiers (example)

Translate the following statement into a logical expression.
“Every real number except zero has a multiplicative inverse.”

A multiplicative inverse of a real number x is a real number y such that
\( xy = 1 \).

Solution:
- Introduce variables
  “For every real number except zero, there is a multiplicative inverse.”
  “For every real number x, if x \( \neq 0 \), then there is a real number y such that xy = 1.”
Nested quantifiers (example)

Translate the following statement into a logical expression.
“Every real number except zero has a multiplicative inverse.”

A multiplicative inverse of a real number $x$ is a real number $y$ such that $xy = 1$.

Solution:
- Translate it to a logical expression
  “For every real number $x$, if $x \neq 0$, then there is a real number $y$ such that $xy = 1$.”

$$\forall x ((x \neq 0) \rightarrow \exists y (xy = 1)) \quad \text{domain: real numbers}$$
Nested quantifiers (example)

Translate the following statement into English.
\( \forall x (C(x) \lor \exists y (C(y) \land F(x,y))) \)

- C(x): x has a computer.
- F(x,y): x and y are friends.
- Domain of x and y: all students

Solution:

“For every student x, x has a computer or there is a student y such that y has a computer and x and y are friends.”

“Every student has a computer or has a friend that has a computer.”
Nested quantifiers (example)

Translate the following statement into English.
\[ \exists x \, \forall y \, \forall z \, ((F(x,y) \land F(x,z) \land (y \neq z)) \rightarrow \neg F(y,z)) \]
- F(x,y): x and y are friends.
- Domain of x, y and z: all students

Solution:
“There is a student x such that for all students y and all students z, if x and y are friends, x and z are friend and z and y are not the same student, then y and z are not friend.”

“There is a student none of whose friends are also friends with each other.”
Nested quantifiers (example)

Translate the following statement into logical expression.

“If a person is a student and is computer science major, then this person takes a course in mathematics.”

Solution:

- Determine individual propositional functions
  - S(x): x is a student.
  - C(x): x is a computer science major.
  - T(x,y): x takes a course y.

- Translate the sentence into logical expression
  \[ \forall x ((S(x) \land C(x)) \rightarrow \exists y T(x,y)) \]

Domain of x: all people
Domain of y: all courses in mathematics
Nested quantifiers (example)

Translate the following statement into logical expression.
“Everyone has exactly one best friend.”

Solution:

- Determine individual propositional function
  - $B(x,y)$: $y$ is the best friend of $x$.

- Express the English statement using variable and individual propositional function
  - For all $x$, there is $y$ who is the best friend of $x$ and for every person $z$, if person $z$ is not person $y$, then $z$ is not the best friend of $x$.

- Translate the sentence into logical expression
  
  $\forall x \, \exists y \, (B(x,y) \land \forall z \, ((z \neq y) \rightarrow \neg B(x,z)))$

  Domain of $x$, $y$ and $z$: all people
Nested quantifiers (example)

Translate the following statement into logical expression. “Everyone has exactly one best friend.”

Solution:

- Determine individual propositional function
  - \( B(x,y) \): y is the best friend of x.

- Express the English statement using variable and individual propositional function
  - For all x, there is y who is the best friend of x and for every person z, if person z is not person y, then z is not the best friend of x.

- Translate the sentence into logical expression
  \( \forall x \exists y \forall z \ ( (B(x,y) \land B(x,z)) \rightarrow (y = z) ) \)
  Domain of x, y and z: all people
Nested quantifiers (example)

Translate the following statement into logical expression.
“**There is a person who has taken a flight on every airline in the world.**”

Solution:

- Determine individual propositional function
  - F(x,f): x has taken flight f.
  - A(f,a): flight f is on airline a.

- Translate the sentence into logical expression
  \[ \exists x \ \forall a \ \exists f \ (F(x,f) \land A(f,a)) \]
  Domain of x: all people
  Domain of f: all flights
  Domain of a: all airlines
Nested quantifiers (example)

Translate the following statement into logical expression.
“There is a person who has taken a flight on every airline in the world.”

Solution:
- Determine individual propositional function
  \( R(x,f,a): x \) has taken flight \( f \) on airline \( a \).
- Translate the sentence into logical expression
  \( \exists x \ \forall a \ \exists f \ R(x,f,a) \)
  Domain of \( x \): all people
  Domain of \( f \): all flights
  Domain of \( a \): all airlines
Negating quantified expressions (review)

\[ \neg \forall x \, P(x) \quad \exists x \, \neg P(x) \]

\[ \neg \exists x \, P(x) \quad \forall x \, \neg P(x) \]
Negating nested quantifiers

- Rules for negating statements involving a single quantifiers can be applied for negating statements involving nested quantifiers.
Negating nested quantifiers
(example)

What is the negation of the following statement?
\( \forall x \ \exists y \ (x = -y) \)

Solution:
\[ \neg \ \forall x \ P(x) \]
\[ P(x) = \exists y \ (x = -y) \]
\[ \exists x \ \neg P(x) \]
\[ \exists x \ (\neg \ \exists y \ (x = -y)) \]
\[ \exists x \ (\forall y \ \neg (x = -y)) \]
\[ \exists x \ \forall y \ (x \neq -y) \]
Negating nested quantifiers (example)

Translate the following statement in logical expression?
“There is not a person who has taken a flight on every airline.”

Solution:

- Translate the positive sentence into logical expression
  - \( \exists x \ \forall a \ \exists f \ (F(x,f) \land A(f,a)) \)  
    by previous example
  - \( F(x,f): x \) has taken flight \( f \). \( A(f,a): \) flight \( f \) is on airline \( a \).

- Find the negation of the logical expression
  - \( \neg \exists x \ \forall a \ \exists f \ (F(x,f) \land A(f,a)) \)
  - \( \forall x \ \neg \forall a \ \exists f \ (F(x,f) \land A(f,a)) \)
  - \( \forall x \ \exists a \ \neg \exists f \ (F(x,f) \land A(f,a)) \)
  - \( \forall x \ \exists a \ \forall f \ \neg (F(x,f) \land A(f,a)) \)
  - \( \forall x \ \exists a \ \forall f \ (\neg F(x,f) \lor \neg A(f,a)) \)
Recommended exercises
1, 3, 10, 13, 23, 25, 27, 33, 39