York University

Faculty of Science and Engineering

MATH 1090: Facts-List for the December 2017 Examination (to be held Dec 7, 2017)

The following are the axioms of Propositional Calculus: In what follows, A, B, C stand for arbitrary formulae.

Properties of $\equiv$

Associativity of $\equiv$	$((A \equiv B) \equiv C) \equiv (A \equiv (B \equiv C))$	(1)
Symmetry of $\equiv$	$(A \equiv B) \equiv (B \equiv A)$	(2)

Properties of $\bot, \top$

$\top$ vs. $\bot$	$\top \equiv \bot \equiv \bot$	(3)

Properties of $\neg$

Introduction of $\neg$	$\neg A \equiv A \equiv \bot$	(4)

Properties of $\vee$

Associativity of $\vee$	$(A \vee B) \vee C \equiv A \vee (B \vee C)$	(5)
Symmetry of $\vee$	$A \vee B \equiv B \vee A$	(6)
Idempotency of $\vee$	$A \vee A \equiv A$	(7)
Distributivity of $\vee$ **over** $\equiv$	$A \vee (B \equiv C) \equiv A \vee B \equiv A \vee C$	(8)
Excluded Middle	$A \vee \neg A$	(9)

Properties of $\wedge$

Golden Rule	$A \wedge B \equiv A \equiv B \equiv A \vee B$	(10)

Properties of $\rightarrow$

Implication	$A \rightarrow B \equiv A \vee B \equiv B$	(11)

The **Primary** Boolean rules are:

$$\frac{A, A \equiv B}{B} \qquad (Eqn)$$

and

$$\frac{A \equiv B}{C[\mathbf{p} := A] \equiv C[\mathbf{p} := B]} \qquad (Leib)$$

The following are the Predicate Calculus Axioms:

Any partial generalisation of any formula in groups* Ax1–Ax6 *is an axiom for Predicate Calculus.

Groups **Ax1–Ax6** contain the following schemata:

Ax1. Every tautology.

Ax2. $(\forall \mathbf{x})A \to A[\mathbf{x} := t]$, for any term t.

Ax3. $A \to (\forall \mathbf{x})A$, provided $\mathbf{x}$ is not free in A.

Ax4. $(\forall \mathbf{x})(A \to B) \to (\forall \mathbf{x})A \to (\forall \mathbf{x})B$.

Ax5. For *each* object variable $\mathbf{x}$, the formula $\mathbf{x} = \mathbf{x}$.

Ax6. For any terms t, s, the schema $t = s \to (A[\mathbf{x} := t] \equiv A[\mathbf{x} := s])$.

The **Primary** First-Order rules are the same as the Boolean; to be exact:

$$\frac{A, A \equiv B}{B} \qquad (Eqn)$$

and

$$\frac{A \equiv B}{C[\mathbf{p} := A] \equiv C[\mathbf{p} := B]}, \text{ where } \mathbf{p} \text{ is not in the scope of any quantifier} \qquad (BL)$$

"BL" stands for "Boolean Leibniz".

The following metatheorems are good for ***both*** Propositional and Predicate Calculus:

1. *Redundant* $\top$. $\Gamma \vdash A$ iff $\Gamma \vdash A \equiv \top$
2. *Modus Ponens (MP).* $A, A \to B \vdash B$
3. *Cut Rule.* $A \vee B, \neg A \vee C \vdash B \vee C$
4. *Deduction Theorem.* If $\Gamma, A \vdash B$, then $\Gamma \vdash A \to B$
5. *Proof by contradiction.* $\Gamma, \neg A \vdash \bot$ iff $\Gamma \vdash A$
6. *Post's Theorem.* (Also called "tautology theorem", or even "completeness of Propositional Calculus theorem")

   If $\models_{\text{taut}} A$, then $\vdash A$.

   Also: If $\Gamma \models_{\text{taut}} A$ for finite Γ, then also $\Gamma \vdash A$.
7. *Proof by cases.* $A \to B, C \to D \vdash A \vee C \to B \vee D$

   Also the special case: $A \to B, C \to B \vdash A \vee C \to B$

The Existential Quantifier ∃

$$(\exists \mathbf{x})A \textit{ stands for } \neg(\forall \mathbf{x})\neg A$$

therefore $(\exists \mathbf{x})A \equiv \neg(\forall \mathbf{x})\neg A$ is a tautology, hence an absolute theorem.

Useful facts from Predicate Calculus (proved in class—you may use them without proof):

We ***know*** that SL and WL (*not stated here; you should know them well!*) are **derived rules** useful in equational proofs within predicate calculus.

- ▶ More "rules" and (meta)theorems.

(i) *"Dummy renaming".*

If $\mathbf{z}$ does not occur in $(\forall \mathbf{x})A$ as either free or bound, then $\vdash (\forall \mathbf{x})A \equiv (\forall \mathbf{z})\big(A[\mathbf{x} := \mathbf{z}]\big)$

If $\mathbf{z}$ does not occur in $(\exists \mathbf{x})A$ as either free or bound, then $\vdash (\exists \mathbf{x})A \equiv (\exists \mathbf{z})\big(A[\mathbf{x} := \mathbf{z}]\big)$

(ii) $\forall$ *over* $\circ$ *distribution, where "*$\circ$*" is "*$\vee$*" or "*$\to$*".*

$$\vdash A \circ (\forall \mathbf{x})B \equiv (\forall \mathbf{x})(A \circ B), \textbf{ provided } \mathbf{x} \text{ is not free in } A$$

$\exists$ *over* $\wedge$ *distribution*

$$\vdash A \wedge (\exists \mathbf{x})B \equiv (\exists \mathbf{x})(A \wedge B), \textbf{ provided } \mathbf{x} \text{ is not free in } A$$

(iii) $\forall$ *over* $\wedge$ *distribution.*

$$\vdash (\forall \mathbf{x})A \wedge (\forall \mathbf{x})B \equiv (\forall \mathbf{x})(A \wedge B)$$

$\exists$ *over* $\vee$ *distribution.*

$$\vdash (\exists \mathbf{x})A \vee (\exists \mathbf{x})B \equiv (\exists \mathbf{x})(A \vee B)$$

(iv) $\forall$ *commutativity (symmetry).*

$$\vdash (\forall \mathbf{x})(\forall \mathbf{y})A \equiv (\forall \mathbf{y})(\forall \mathbf{x})A$$

(v) *Specialisation. "Spec"* $(\forall \mathbf{x})A \vdash A[\mathbf{x} := t]$, for any term t.

Dual of Specialisation. "Dual Spec" $A[\mathbf{x} := t] \vdash (\exists \mathbf{x})A$, for any term t.

(vi) *Generalisation. "Gen"* If $\Gamma \vdash A$ and if, moreover, the formulae in Γ have **no free x occurrences**, then also $\Gamma \vdash (\forall \mathbf{x})A$.

(vii) $\forall$ *Monotonicity.* If $\Gamma \vdash A \to B$ so that the formulae in Γ have **no free x occurrences**, then we can infer

$$\Gamma \vdash (\forall \mathbf{x})A \to (\forall \mathbf{x})B$$

(viii) $\forall$ *Introduction; a special case of* $\forall$ *Monotonicity.* If $\Gamma \vdash A \to B$ so that neither the formulae in Γ nor A have **any free x occurrences**, then we can infer

$$\Gamma \vdash A \to (\forall \mathbf{x})B$$

(ix) Finally, the *Auxiliary Variable ("witness") Metatheorem.* If $\Gamma \vdash (\exists \mathbf{x})A$, and if $\mathbf{y}$ is a variable that ***does not*** occur as either free or bound variable in any of $(\exists \mathbf{x})A$ or B or the formulae of Γ, then

$$\Gamma, A[\mathbf{x} := \mathbf{y}] \vdash B \text{ implies } \Gamma \vdash B$$

Semantics facts

Propositional Calculus	Predicate Calculus
(Boolean Soundness) $\vdash A$ implies $\models_{\text{taut}} A$	$\vdash A$ does **NOT** imply $\models_{\text{taut}} A$
(Post) $\models_{\text{taut}} A$ implies $\vdash A$	However, (Post) $\models_{\text{taut}} A$ implies $\vdash A$
	(Pred. Calc. Soundness) $\vdash A$ implies $\models A$

CAUTION! The above facts/tools are only a fraction of what we have covered in class. They are ***very important and very useful***, and that is why they are listed for your reference here.

You can also use *without proof* ***ALL*** the things we have covered (such as the absolute theorems known as "$\exists$-definition", "de Morgan's laws", etc.).

But these —the unlisted ones— are up to you to remember and to correctly state!

Whenever in doubt of whether or not a "tool" you are about to use was indeed covered in class, ***prove*** *the validity/fitness of the tool before using it!*