York University

Faculty of Pure and Applied Science, Faculty of Arts, Atkinson Faculty

MATH 1090: Facts-List for the December 2005 Examination

Sections A (Tourlakis) and B (Farah)

NAME (**print in ink**):______________________________
(Family) (First)

SIGNATURE (**in ink**):______________________________

SECTION (**in ink**): ☐ STUDENT NUMBER(**in ink**) _______________

This handout must be filled in (name, etc.) as required above, and handed in along with your exam paper. <u>It is NOT allowed to write anything on these pages beyond the information we are asking above.</u>

Here is a list of some tools/facts covered in MATH1090, <u>which you may use "off the shelf", without proof.</u>

<u>The following are the axioms of Propositional Calculus:</u> In what follows, A, B, C stand for arbitrary formulae.

<u>Properties of $\equiv$</u>

Associativity of $\equiv$ $((A \equiv B) \equiv C) \equiv (A \equiv (B \equiv C))$ (1)

Symmetry of $\equiv$ $(A \equiv B) \equiv (B \equiv A)$ (2)

<u>Properties of *false*, *true*</u>

true **vs.** *false* $\mathit{true} \equiv \mathit{false} \equiv \mathit{false}$ (3)

<u>Properties of $\neg$</u>

Introduction of $\neg$ $\neg A \equiv A \equiv \mathit{false}$ (4)

<u>Properties of $\vee$</u>

Associativity of $\vee$	$(A \vee B) \vee C \equiv A \vee (B \vee C)$	(5)
Symmetry of $\vee$	$A \vee B \equiv B \vee A$	(6)
Idempotency of $\vee$	$A \vee A \equiv A$	(7)
Distributivity of $\vee$ **over** $\equiv$	$A \vee (B \equiv C) \equiv A \vee B \equiv A \vee C$	(8)
Excluded Middle	$A \vee \neg A$	(9)

Properties of $\wedge$

Golden Rule	$A \wedge B \equiv A \equiv B \equiv A \vee B$	(10)

Properties of $\Rightarrow$

Implication	$A \Rightarrow B \equiv A \vee B \equiv B$	(11)

The **Primary** Boolean rules are:

$$\frac{A, A \equiv B}{B} \tag{Eqn}$$

and

$$\frac{A \equiv B}{C[p := A] \equiv C[p := B]} \tag{$Leib$}$$

The following are the Predicate Calculus Axioms:

Any partial generalisation of any formula in groups* Ax1–Ax6 *is an axiom for Predicate Calculus.

Groups **Ax1–Ax6** contain:

Ax1. The schemata (1)–(11) above (from the Boolean case).

Ax2. For every formula A, $(\forall x)A \Rightarrow A[x := t]$, for any term t.

Ax3. For every formula A and variable x *not free in* A, the formula $A \Rightarrow (\forall x)A$.

Ax4. For every formulae A and B, $(\forall x)(A \Rightarrow B) \Rightarrow (\forall x)A \Rightarrow (\forall x)B$.

Ax5. For *each* object variable x, the formula $x = x$.

Ax6. For any formula A, any object variable x and any terms t, s, the formula $t = s \Rightarrow (A[x := t] \equiv A[x := s])$.

Primary rules of inference are **the same as for the Boolean Case**, namely, **Equanimity** and **Leibniz**. The fact that Leibniz is applied to the Boolean abstraction of formulae is reflected by its name —Boolean Leibniz or "BL"— and the restriction on where p may occur (see below).

$$\frac{A \equiv B}{C[p := A] \equiv C[p := B]}, \text{ provided } p \text{ is } \textbf{not} \text{ in the scope of a quantifier.} \tag{BL}$$

The following metatheorems are good for ***both*** Propositional and Predicate Calculus:

1. *Redundant True.* $\Gamma \vdash A$ iff $\Gamma \vdash A \equiv true$
2. *Modus Ponens (MP).* $A, A \Rightarrow B \vdash B$
3. *Cut Rule.* $A \vee B, \neg A \vee C \vdash B \vee C$
4. *Deduction Theorem.* If $\Gamma, A \vdash B$, then $\Gamma \vdash A \Rightarrow B$
5. *Proof by contradiction.* $\Gamma, \neg A \vdash false$ iff $\Gamma \vdash A$
6. *Post's Theorem.* (Also called "tautology theorem", or even "completeness of Propositional Calculus theorem")

   If $\models_{\text{taut}} A$, then $\vdash A$.

   Also: If $\Gamma \models_{\text{taut}} A$, then $\Gamma \vdash A$.
7. *Proof by cases.* $A \Rightarrow B$, $C \Rightarrow D \vdash A \vee C \Rightarrow B \vee D$

   Also the special case: $A \Rightarrow B$, $C \Rightarrow B \vdash A \vee C \Rightarrow B$

Translations

$(\exists x)A$ translates to $\neg(\forall x)\neg A$

$(\forall x|A : B)$ translates to $(\forall x)(A \Rightarrow B)$ ("Range trading with $\forall$")

$(\exists x|A : B)$ translates to $(\exists x)(A \wedge B)$ ("Range trading with $\exists$")

Useful facts from Predicate Calculus (proved in class—you may use them without proof):

We ***know*** that SL and WL (as well as GS-Leibniz "8.12(a)" and "8.12(b)") are **derived rules**. These are the following ("GS"-notation is used in 8.12(a–b) below):

$$\text{Same as BL, without the condition: } A \equiv B \vdash C[p := A] \equiv C[p := B] \qquad (SL)$$

$$\text{if } \vdash A \equiv B, \text{ then } \vdash C[p \setminus A] \equiv C[p \setminus B] \qquad (WL)$$

$$\text{if } \vdash A \equiv B, \text{ then } \vdash \big(* x|C[p := A] : D\big) \equiv \big(* x|C[p := B] : D\big) \qquad (8.12(a))$$

$$\text{if } \vdash D \Rightarrow (A \equiv B), \text{ then } \vdash \big(* x|D : C[p := A]\big) \equiv \big(* x|D : C[p := B]\big) \qquad (8.12(b))$$

where in 8.12(a–b) "$*$" stands everywhere for the symbol "$\forall$", or the symbol "$\exists$".

► More "rules" and (meta)theorems.

(i)

$$\vdash A \equiv (\forall x)A, \textbf{ provided } x \text{ is not free in } A$$

$$\vdash A \equiv (\exists x)A, \textbf{ provided } x \text{ is not free in } A$$

(ii) *Dummy renaming.*

If z does not occur in $(\forall x)A$ as either free or bound, then $\vdash (\forall x)A \equiv (\forall z)\big(A[x := z]\big)$

If z does not occur in $(\exists x)A$ as either free or bound, then $\vdash (\exists x)A \equiv (\exists z)\big(A[x := z]\big)$

(iii) $\forall$ *over* $\circ$ *distribution, where "*$\circ$*" is "*$\vee$*" or "*$\Rightarrow$*".*

$$\vdash A \circ (\forall x)B \equiv (\forall x)(A \circ B), \textbf{ provided } x \text{ is not free in } A$$

$\exists$ *over* $\wedge$ *distribution*

$$\vdash A \wedge (\exists x)B \equiv (\exists x)(A \wedge B), \textbf{ provided } x \text{ is not free in } A$$

(iv) $\forall$ *over* $\wedge$ *distribution.*

$$\vdash (\forall x)A \wedge (\forall x)B \equiv (\forall x)(A \wedge B)$$

$\exists$ *over* $\vee$ *distribution.*

$$\vdash (\exists x)A \vee (\exists x)B \equiv (\exists x)(A \vee B)$$

(v) $\forall$ *commutativity (symmetry).*

$$\vdash (\forall x)(\forall y)A \equiv (\forall y)(\forall x)A$$

$\exists$ *commutativity (symmetry).*

$$\vdash (\exists x)(\exists y)A \equiv (\exists y)(\exists x)A$$

(vi) *Specialisation.* $(\forall x)A \vdash A[x := t]$, for any term t.

Dual of Specialisation. $A[x := t] \vdash (\exists x)A$, for any term t.

(vii) *Generalisation.* If $\Gamma \vdash A$ and if, moreover, the formulae in Γ have **no free x occurrences**, then also $\Gamma \vdash (\forall x)A$.

(viii) $\forall$ *Monotonicity.* If $\Gamma \vdash A \Rightarrow B$ so that the formulae in Γ have **no free x occurrences**, then we can infer

$$\Gamma \vdash (\forall x)A \Rightarrow (\forall x)B$$

$\exists$ *Monotonicity.* If $\Gamma \vdash A \Rightarrow B$ so that the formulae in Γ have **no free x occurrences**, then we can infer

$$\Gamma \vdash (\exists x)A \Rightarrow (\exists x)B$$

(ix) $\forall$ *Introduction; a special case of* $\forall$ *Monotonicity that uses (i) above.* If $\Gamma \vdash A \Rightarrow B$ so that neither the formulae in Γ nor A have **any free** x **occurrences**, then we can infer

$$\Gamma \vdash A \Rightarrow (\forall x)B$$

$\exists$ *Introduction; a special case of* $\exists$ *Monotonicity that uses (i) above.* If $\Gamma \vdash A \Rightarrow B$ so that neither the formulae in Γ nor B have **any free** x **occurrences**, then we can infer

$$\Gamma \vdash (\exists x)A \Rightarrow B$$

(x) (*Equals-for-equals in terms*) For any terms t, s, t' and variable x,

$$\vdash t = t' \Rightarrow s[x := t] = s[x := t']$$

(xi) Finally, the *Auxiliary Variable ("witness") Metatheorem.* If $\Gamma \vdash (\exists x)A$, and if y is a variable that ***does not*** occur as either free or bound variable in any of A or B or the formulae of Γ, then

$$\Gamma, A[x := y] \vdash B \text{ implies } \Gamma \vdash B$$

Semantics facts

Propositional Calculus	Predicate Calculus
(Boolean Soundness) $\vdash A$ implies $\models_{\text{taut}} A$	$\vdash A$ does **NOT** imply $\models_{\text{taut}} A$
(Post) $\models_{\text{taut}} A$ implies $\vdash A$	However, (Post) $\models_{\text{taut}} A$ implies $\vdash A$
	(Pred. Calc. Soundness) $\vdash A$ implies $\models A$
	(Gödel Completeness) $\models A$ implies $\vdash A$

CAUTION! The above facts/tools are only a fraction of what we have covered in class. They are ***very important and very useful***, and that is why they are listed for your reference here.

You can also use *without proof* ***ALL*** the things we have covered (such as "one-point-rule", "de Morgan's laws", etc.).

But these —the unlisted ones— are up to you to remember and to correctly state!

Whenever in doubt of whether or not a "tool" you are about to use was covered in class, ***prove*** *the validity/fitness of the tool before using it!*