# Lecture #9, Oct. 7 —Continued

**0.0.1 Theorem.** $\vdash \top \equiv \neg\bot$

**Proof.** (Equational)

$$
\begin{array}{ll}
 & \top \\
\Leftrightarrow & \langle \text{axiom} \rangle \\
 & \bot \equiv \bot \\
\Leftrightarrow & \langle \text{axiom} \rangle \\
 & \neg\bot
\end{array}
$$

□

**0.0.2 Theorem.** $\vdash \bot \equiv \neg\top$

**Proof.** (Equational)

$$
\begin{array}{ll}
 & \neg\top \\
\Leftrightarrow & \langle \text{axiom} \rangle \\
 & \top \equiv \bot \\
\Leftrightarrow & \langle \text{red. } \top \rangle \\
 & \bot
\end{array}
$$

□

**0.0.3 Theorem.** $\vdash A \vee \top$

**Proof.**

$$
\begin{aligned}
& A \vee \top \\
\Leftrightarrow\ & \langle (\mathit{Leib}) + \text{axiom: } \top \equiv \bot \equiv \bot\text{; “Denom:” } A \vee \mathbf{p}\text{; } \textbf{Mind the brackets!} \rangle \\
& A \vee (\bot \equiv \bot) \\
\Leftrightarrow\ & \langle \text{axiom} \rangle \\
& A \vee \bot \equiv A \vee \bot \quad \text{Bingo!}
\end{aligned}
$$

□

Recall about $\equiv$ that, by axiom (1) and a theorem we proved in class, if we read Axiom (1) right to left we get a theorem.

This meant that in a chain of TWO "$\equiv$" it does not matter how we inserted brackets, and thus can totally omit them.

Without proof we stated that in a chain of any number of $\equiv$ we may omit brackets. See course URL

`http://www.cs.yorku.ca/~gt/courses/MATH1090F20/1090.html`

bullet three under "Misc Notes".

The same holds for a chain of $\vee$ using the same kind of steps:

- Prove that $\vdash A \vee (B \vee C) \equiv (A \vee B) \vee C$, that is, reading axiom (5) right to left is a theorem.
  Indeed here is a short Hilbert proof:

  (1) $(A \vee B) \vee C \equiv A \vee (B \vee C)$ ⟨axiom (5)⟩
  (2) $A \vee (B \vee C) \equiv (A \vee B) \vee C$ ⟨1 + Rule *comm.*⟩

  Thus, in a chain of TWO "$\vee$" it does not matter how we inserted brackets, and thus can totally omit them.

- The NOTE in the URL above generalises this to a chain of any number of "$\vee$".

OK, so we <u>do not need to</u> show bracketing in a chain of $\equiv$ or $\vee$.

How about moving formulas around in such a chain? (Permuting them).

It is OK! I prove this for $\vee$-chains. The proof is identical for $\equiv$-chains.

EXERCISE!!

Start with this theorem:

$$\vdash B \vee C \vee D \equiv D \vee C \vee B$$

<u>Indeed here is a proof</u>:

$$\begin{aligned} & B \vee C \vee D \\ \Leftrightarrow\ & \langle \text{axiom (6)} \rangle \\ & D \vee B \vee C \qquad\qquad (*) \\ \Leftrightarrow\ & \langle (Leib) + \text{axiom (6). "Denom:" } D \vee \mathbf{p} \rangle \\ & D \vee C \vee B \end{aligned}$$

More generally we CAN DO an arbitrary swap (not only the END-FORMULAS), that is, we have the theorem

$$\vdash A \vee B \vee C \vee D \vee E \equiv A \vee D \vee C \vee B \vee E$$

Follows by an application of the previous special case:

$$\begin{array}{ll} & A \vee B \vee C \vee D \vee E \\ \Leftrightarrow & \langle (Leib) + \text{special case. “Denom:”}\ A \vee \mathbf{p} \vee E \rangle \\ & A \vee D \vee C \vee B \vee E \end{array}$$

**0.0.4 Theorem.** $\vdash A \vee \bot \equiv A$

**Proofs.** (Equational)

This time we work with the entire formula, not just one of the sides of "$\equiv$".

How do we know? We don't. It is just practice.

$$
\begin{array}{ll}
 & A \vee \bot \equiv A \\
\Leftrightarrow & \langle (Leib) + \text{axiom } A \equiv A \vee A;\ \text{“Denom:”}\ \ A \vee \bot \equiv \mathbf{p} \rangle \\
 & A \vee \bot \equiv A \vee A \\
\Leftrightarrow & \langle \text{axiom } \vee \text{ over } \equiv \rangle \\
 & A \vee (\bot \equiv A) \\
\Leftrightarrow & \langle (Leib) + \text{axiom: } \bot \equiv A \equiv \neg A;\ \text{“Denom:”}\ \ A \vee \mathbf{p} \rangle \\
 & A \vee \neg A
\end{array}
$$

*Bingo*! □

**0.0.5 Theorem.** $\vdash A \to B \equiv \neg A \vee B$

**Proof.**

$$
\begin{aligned}
& A \to B \\
\Leftrightarrow\ & \langle \text{axiom} \rangle \\
& A \vee B \equiv B \qquad\qquad \text{HERE} \\
\Leftrightarrow\ & \langle (\mathit{Leib}) + 0.0.4;\ \text{“Denom:”}\ A \vee B \equiv \mathbf{p} \rangle \\
& A \vee B \equiv \bot \vee B \\
\Leftrightarrow\ & \langle \text{axiom} \rangle \\
& (A \equiv \bot) \vee B \\
\Leftrightarrow\ & \langle (\mathit{Leib}) + \text{axiom};\ \text{“Denom:”}\ \mathbf{p} \vee B \rangle \\
& \neg A \vee B
\end{aligned}
$$

□

**Comment on “same” p in above proof.**

**0.0.6 Corollary.** $\vdash \neg A \vee B \equiv A \vee B \equiv B$

**Proof.** Start the above proof from “HERE”. □

**0.0.7 Theorem. (de Morgan 1)** $\vdash A \wedge B \equiv \neg(\neg A \vee \neg B)$

**Proof.**

Long but obvious. Start with the most complex side!

$$
\begin{aligned}
& \neg(\neg A \vee \neg B)\\
\Leftrightarrow\ & \langle \text{axiom} \rangle\\
& \neg A \vee \neg B \equiv \bot\\
\Leftrightarrow\ & \langle (Leib) + 0.0.6;\ \text{“Denom:”}\ \mathbf{p} \equiv \bot \rangle\\
& A \vee \neg B \equiv \neg B \equiv \bot\\
\Leftrightarrow\ & \langle (Leib) + \text{axiom};\ \text{“Denom:”}\ A \vee \neg B \equiv \mathbf{p} \text{ —\underline{order} does not matter!} \rangle\\
& A \vee \neg B \equiv B\\
\Leftrightarrow\ & \langle (Leib) + 0.0.6;\ \text{“Denom:”}\ \mathbf{p} \equiv B \rangle\\
& A \vee B \equiv A \equiv B\\
\Leftrightarrow\ & \langle \text{GR axiom —\underline{order} does not matter} \rangle\\
& A \wedge B
\end{aligned}
$$

□

**0.0.8 Corollary. (de Morgan 2)** $\vdash A \vee B \equiv \neg(\neg A \wedge \neg B)$

**Proof.** See Text. Better still, EXERCISE!

## *About "$\wedge$"*

**0.0.9 Theorem.** $\vdash A \wedge A \equiv A$

**Proof.**

$$\begin{aligned} & A \wedge A \equiv A \\ \Leftrightarrow & \langle \text{axiom —}\underline{\text{order}} \text{ does not matter} \rangle \\ & A \vee A \equiv A \qquad \textit{Bingo!} \end{aligned}$$

□

**0.0.10 Theorem.** $\vdash A \wedge \top \equiv A$

**Proof.**

$$\begin{aligned} & A \wedge \top \equiv A \\ \Leftrightarrow & \langle \text{GR axiom} \rangle \\ & A \vee \top \equiv \top \\ \Leftrightarrow & \langle \text{Red. } \top \rangle \\ & A \vee \top \qquad \textit{Bingo!} \end{aligned}$$

□

**0.0.11 Theorem.** $\vdash A \wedge \bot \equiv \bot$

**Proof.**

$$\begin{aligned} & A \wedge \bot \equiv \bot \\ \Leftrightarrow & \langle \text{GR axiom} \rangle \\ & A \vee \bot \equiv A \quad \textit{Bingo!} \end{aligned}$$

□

**0.0.12 Theorem.**

(*i*) $\vdash A \vee B \wedge C \equiv (A \vee B) \wedge (A \vee C)$

*and*

(*ii*) $\vdash A \wedge (B \vee C) \equiv A \wedge B \vee A \wedge C$

The above are written in least parenthesised notation!

**Proof.**

We just prove (*i*).

$$
\begin{aligned}
& (A \vee B) \wedge (A \vee C) \\
\Leftrightarrow\ & \langle \text{GR} \rangle \\
& A \vee B \vee A \vee C \equiv A \vee B \equiv A \vee C \\
\Leftrightarrow\ & \langle (Leib) + \text{scramble an } \vee\text{-chain; “Denom:” } \mathbf{p} \equiv A \vee B \equiv A \vee C \rangle \\
& A \vee A \vee B \vee C \equiv A \vee B \equiv A \vee C \\
\Leftrightarrow\ & \langle (Leib) + \text{axiom; “Denom:” } \mathbf{p} \vee B \vee C \equiv A \vee B \equiv A \vee C \rangle \\
& A \vee B \vee C \equiv A \vee B \equiv A \vee C
\end{aligned}
$$

HERE WE STOP, and try to reach this result from the other side: $A \vee B \wedge C$.

$$
\begin{aligned}
& A \vee B \wedge C \\
\Leftrightarrow\ & \langle (\mathit{Leib}) + \text{GR; “Denom:” } A \vee \mathbf{p}\text{; mind brackets!} \rangle \\
& A \vee (B \vee C \equiv B \equiv C) \\
\Leftrightarrow\ & \langle \text{axiom} \rangle \\
& A \vee B \vee C \equiv A \vee (B \equiv C) \\
\Leftrightarrow\ & \langle (\mathit{Leib}) + \text{axiom; “Denom:” } A \vee B \vee C \equiv \mathbf{p} \rangle \\
& A \vee B \vee C \equiv A \vee B \equiv A \vee C
\end{aligned}
$$

**0.0.13 Corollary.** $\vdash A \vee B \to C \equiv (A \to C) \wedge (B \to C)$

**Proof.**

$$
\begin{aligned}
& A \vee B \to C \\
\Leftrightarrow\ & \langle 0.0.5 \rangle \\
& \neg(A \vee B) \vee C \\
\Leftrightarrow\ & \langle (\textit{Leib}) + 0.0.8;\ \text{"Denom:"}\ \neg\mathbf{p} \vee C \rangle \\
& \neg\neg(\neg A \wedge \neg B) \vee C \\
\Leftrightarrow\ & \langle (\textit{Leib}) + \text{double neg.; "Denom:"}\ \mathbf{p} \vee C \rangle \\
& (\neg A \wedge \neg B) \vee C \\
\Leftrightarrow\ & \langle 0.0.12 \rangle \\
& (\neg A \vee C) \wedge (\neg B \vee C) \\
\Leftrightarrow\ & \langle \text{obvious } (\textit{Leib}), \text{ twice} + 0.0.5 \rangle \\
& (A \to C) \wedge (B \to C)
\end{aligned}
$$

□

## Lecture #10, Oct. 9

**0.0.14 Corollary.** $\vdash A \to B \wedge C \equiv (A \to B) \wedge (A \to C)$

**Proof.**

$$
\begin{aligned}
& A \to B \wedge C \\
\Leftrightarrow\ & \langle 0.0.5 \rangle \\
& \neg A \vee B \wedge C \\
\Leftrightarrow\ & \langle 0.0.12 \rangle \\
& (\neg A \vee B) \wedge (\neg A \vee C) \\
\Leftrightarrow\ & \langle \text{obvious } (\mathit{Leib}) \text{ twice} + 0.0.5 \rangle \\
& (A \to B) \wedge (A \to C)
\end{aligned}
$$

□

Until now we only proved absolute theorems Equationally.

How about theorems with HYPOTHESES?

To do so we use the Redundant $\top$ METAtheorem:

$\Gamma \vdash A$ iff $\Gamma \vdash A \equiv \top$

The Technique is demonstrated via Examples!

**0.0.15 Example.** (1) $A, B \vdash A \wedge B$

(2) $A \vee A \vdash A$

(3) $A \vdash A \vee B$

(4) $A \wedge B \vdash A$

For (1):

$$
\begin{aligned}
& A \wedge B \\
\Leftrightarrow & \langle (Leib) + \text{hyp } B + \text{Red. } \top \text{ META; “Denom:” } A \wedge \mathbf{p} \rangle \\
& A \wedge \top \\
\Leftrightarrow & \langle 0.0.10 \rangle \\
& A
\end{aligned}
$$

Bingo!

$A, B \vdash B$. Hence $A, B \vdash B \equiv \top$

For (2):

$$\begin{aligned} & A \\ \Leftrightarrow\ & \langle \text{axiom} \rangle \\ & A \vee A \end{aligned}$$

For (3):

$$\begin{aligned} & A \vee B \\ \Leftrightarrow\ & \langle (\mathit{Leib}) + \text{Hyp } A + \text{Red-}\top\text{-META; "Denom:" } \mathbf{p} \vee B \rangle \\ & \top \vee B \qquad \langle \text{Bingo!} \rangle \end{aligned}$$

(4) is a bit trickier:

$$
\begin{aligned}
& A \\
\Leftrightarrow\ & \langle 0.0.10 \rangle \\
& A \wedge \top \\
\Leftrightarrow\ & \langle (\textit{Leib}) + \text{Hyp } A \wedge B + \text{Red-}\top\text{-META; ``Denom:'' } A \wedge \mathbf{p} \rangle \\
& A \wedge A \wedge B \\
\Leftrightarrow\ & \langle (\textit{Leib}) + 0.0.9\text{; ``Denom:'' } \mathbf{p} \wedge B \rangle \\
& A \wedge B \qquad \text{Bingo!}
\end{aligned}
$$

□