COSC 4111/5111 Advising Note —Winter 2013

Here are some dances we have to engage in in order to prove certain functions and predicates are in $\mathcal{E}^0, \mathcal{E}^0_*$.

0.0.0.1 Lemma. *The following functions and predicates are in* $\mathcal{E}^0, \mathcal{E}^0_*$.

(1) $\widetilde{sw}(x, y) =$ *if* $x = 0$ *then* $y + 1$ *else* y

(2) $rem(x, y)$

(3) $x|y$ *(*x *divides* y*)*

(4) $\lfloor x/y \rfloor$

(5) $z = x + y$

(6) $z = xy$

(7) $Pr(x)$

(8) $Seq(x)$

(9) $\pi(x)$

(10) $y = p_n$

(11) $lh(z)$

(12) $\Omega(n, z)$*, meaning, "*$z = p_n^k$*, for some* $k \geq 0$*"*

(13) $\Pi(n, z)$*, meaning, "the* number *of powers* p_n^k *that are* $\leq z$*"*

(14) $z = p_x^y$

(15) $\exp(x, y)$

(16) $(x)_y$

Proof.

(1) $\widetilde{sw}(x, y) =$ if $x = 0$ then $y + 1$ else y:

$\widetilde{sw}(0, y) = y + 1$, $\widetilde{sw}(x + 1, y) = y$, and $\widetilde{sw}(x, y) \leq y + 1$

(2) $rem(x, y)$:

$$\begin{aligned} rem(0, y) &= 0 \\ rem(x+1, y) &= \begin{cases} 0 & \text{if } rem(x, y) = y \dot{-} 1 \\ rem(x, y) + 1 & \text{oth} \end{cases} \\ rem(x, y) &< y \end{aligned}$$

(3) $x|y$ (x divides y):

$$x|y \equiv rem(y, x) = 0$$

(4) $\lfloor x/y \rfloor$:

$$\begin{aligned} \lfloor 0/y \rfloor &= 0 \\ \lfloor x+1/y \rfloor &= \begin{cases} \lfloor x/y \rfloor + 1 & \text{if } rem(x, y) = y \dot{-} 1 \\ \lfloor x/y \rfloor & \text{oth} \end{cases} \\ \lfloor x/y \rfloor &\leq x \end{aligned}$$

(5) $z = x + y$:

$$z = x + y \equiv z \dot{-} x = y \wedge y > 0 \vee y = 0 \wedge z = x$$

(6) $z = xy$:

$$z = xy \equiv z = 0 \wedge (x = 0 \vee y = 0) \vee x > 0 \wedge y > 0 \wedge x|z \wedge \lfloor z/x \rfloor = y$$

(7) $Pr(x)$:

$$Pr(x) \equiv x > 1 \wedge (\forall y)_{\leq x}(y > 0 \wedge y|x \rightarrow y = 1 \vee y = x)$$

(8) $Seq(x)$:

$$Seq(x) \equiv x > 1 \wedge (\forall y, z)_{\leq x}(Pr(y) \wedge Pr(z) \wedge y < z \wedge z|x \rightarrow y|x)$$

(9) $\pi(x)$:

$$\begin{aligned} \pi(0) &= 0 \\ \pi(x+1) &= \widetilde{sw}\big(\chi_{Pr}(x+1), \pi(x)\big) \\ \pi(x) &\leq x \end{aligned}$$

(10) $y = p_n$:

$$y = p_n \equiv Pr(y) \wedge \pi(y) = y + 1$$

(11) $lh(z)$:

$lh(z) = (\mathring{\mu} y)_{\leq z} \neg p_y | z.$

Wait! $\lambda y.p_y$ is too big to be in $\mathcal{E}^0$. Am I allowed to plug it into a variable of an $\mathcal{E}^0_*$ predicate —here $x|z$— and expect an $\mathcal{E}^0_*$ result?

Well, no! But on a *case by case manner* there may be a "clever" way to show it is OK:

$p_y | z \equiv (\exists w)_{\leq z}(w = p_y \wedge w|z)$. "One-point rule" of MATH1090, albeit with a bounded quantifier.

(12) $\Omega(n, z)$:

$\Omega(n, z) \equiv z = 1 \vee (\forall y)_{\leq z}(y > 1 \wedge y|z \to p_n|y)$

(13) $\Pi(n, z)$:

$$\begin{aligned} \Pi(n, 0) &= 0 \\ \Pi(n, z+1) &= \widetilde{sw}\big(\chi_\Omega(n, z+1), \Pi(n, z)\big) \\ \Pi(n, z) &\leq z + 1 \end{aligned}$$

(14) $z = p_x^y$:

$z = p_x^y \equiv \Omega(x, z) \wedge \Pi(x, z) = y + 1$

(15) $\exp(x, y)$:

$\exp(x, y) = (\mathring{\mu} z)_{\leq y} \neg p_x^{z+1} | y$

Noting that $p_x^{z+1}|y \equiv (\exists w)_{\leq y}(w|y \wedge w = p_x^{z+1})$, we are done by (14).

(16) $(x)_y$:

$(x)_y = \exp(y, x) \dot{-} 1$ $\square$