Abstract Thinking,
Intro to the Theory of Computation

SC/COSC 2001 3.0 Lecture Notes

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Contents

Chapter 1

Preface

These notes are by no means complete. They are just those that I have managed to scratch together so far.

Please help me improve them.

Intro:

Thinking Abstractly: On one level, this course can be viewed as teaching a sequence of simple models of computation and simple algorithms. On a higher level, the goal of the course is to teach you to think abstractly about computation. The more abstractions a person has from which to view the problem, the deeper his understanding of it will be, the more tools he will have at his disposal, and the better prepared he will be to design his own innovative ways to solve the problems that may arise in other courses or in the work place. Hence, we present a number of different notations, analogies, and paradigms within which to develop and to think about algorithms.

Explaining: To be able to prove yourself within a test or the world, you need to be able to explain the material well. In addition, explaining it to someone else is the best way to learn it yourself. Hence, I highly recommend spending a lot of time explain the material over and over again out loud to yourself, to each other, and to your stuffed bear.

Dreaming: When designing an algorithm, the tendency is to start coding. When studying, the tendency is to read or to do problems. Though these are important, I would like to emphasis the importance of thinking, even day dreaming, about the material. This can be done while going along with your day, while swimming, showering, cooking, or laying in bed. Ask questions. Why is it done this way and not that way? Invent other algorithms for solving a problem. Then look for input instances for which your algorithm gives the wrong answer. Mathematics is not all linear thinking. If the essence of the material, what the questions are really asking, is allowed to seep down into your subconscious then with time little thoughts will begin to percolate up. Pursue these ideas. Sometimes even flashes of inspiration appear.

Course Outline:

Useful Models: We learn about regular languages, deterministic finite automata, context free grammars, Turing machines, and NP-completeness. We learn the definitions, how to construct, and how to manipulate them.

In addition to teaching us to think abstractly, these models of computation are very useful in themselves.

Regular Languages for describing simple string structures;

Deterministic Finite Automata for understanding simple machines and processes;

Context Free Grammars for describing languages such a English or Java and for describing recursive structures;

Turing Machines for understanding the bare bones of computation and for proving that certain computational problems are not computable;

NP-Completeness for understanding that large classes of useful computational problems require too much time to practically compute.

Describing Algorithms: Learning about different models of computation will help you. Each model provides a different notation for describing algorithms. Each provides a different abstract frame work for thinking about algorithms.

The first model we learn is DFA. Because these perform simplistic tasks, like determining if the number of as in a string is even or odd, students think that they are not useful. However, it is their simplicity that makes then useful. In themselves, they have many applications. However, a main reason for teaching them is that students find it very difficult to think abstractly about computation. Hence, it is good to start with simple ideas.

Compiling Between Models: You will be taught algorithms for mechanically "compiling" an algorithm developed in one model into an equivalent algorithm described in another model.

Iterative Loop Invariant Model $\Longleftrightarrow$ Deterministic Automata: Loop Invariants provide a useful level of abstraction for developing, understanding, and describing iterative algorithms. Deterministic Automata provides another. It is useful and insightful to be able to translate between them.

Deterministic Finite Automata (DFA) $\Longleftrightarrow$ Regular Expressions: As said above, both are very useful levels of abstraction. Hence, it is important to be able to translate between them.

Deterministic Finite Automata (DFA) $\Longleftrightarrow$ Nondeterministic Deterministic Automata (NFA): Another very useful level of abstraction is *nondeterminism*. This can be viewed as *help from above*. The idea is to write your algorithm assuming that you have some very powerful help from your favorite *higher power*. With this help, designing the algorithm is much easier. Once a correct algorithm has been designed within this level of abstraction, the next completely separate task is to *move between* the different levels of abstraction, by mechanically "compiling" your algorithm into an algorithm that does not require help from above.

Nondeterministic Machine: The *transition function* of these machines does not provide only one next state that the machine must be in based on its current configuration. Instead it gives a set of possible states. The result is each input instance does not have just one computation path, but has many. We say that the input instance is *accepted* if there exists at least one computation path that leads to an accept state.

Help From A Higher Power: One can view a computation on a nondeterministic machine as requiring nondeterministic help from a higher power. This power tells the computation the sequence of choices to make to lead it along one of the accepting computation paths to an accepting state. A key feature of this help is that a corrupt higher power can never convince you that there is an accepting computation when there is not one. The reason is that you are able to check that the sequence of choices given to you does in fact lead to an accept state.

Context Free Grammars $\Longleftrightarrow$ Push Down Automata: Context free grammars are very useful. Push down automata are only of passing interest.

Turing Machines $\Longleftrightarrow$ Multi Tapes & Heads $\Longleftrightarrow$ Random Access $\Longleftrightarrow$ JAVA Code: The Church/Turing thesis states that all reasonable models of computation are equivalent as far as which computational problems are computable. This is worth understanding.

Resources Needed for Computation: Suppose your boss wants you to design an algorithm for solving some new computational problem. You should be able to determine what resources this problem requires. Could it be hard wired into a small chip or does it require more computation time and more memory? Can it be solved quickly or will it require more time than the number of atoms in the universe or is it provably unsolvable even given unbounded time.

Learning about different models of computation will help you with this as well. Each model also places restrictions on the resources and types of actions that an algorithm is an is not allowed to do. Then we learn to classify computational problems based on which resources they require.

In this course, you will learn each of these separate levels, as well as ways of moving between the levels. Individually, no level is very difficult. The key to the entire course is to apply these abstract thinking skills to solving new problems that arise in other courses and in the workplace.

Chapter 2

Computational Problems and Models of Computation

The course and the text start by formally defining Deterministic Finite Automaton and Turing machines. My feeling is that students never get the intuition to why this is a natural definition. Hence, this document goes on at length about the intuition behind formal models of computation. I hope it helps.

2.1 History of Computability

- Euclid (a Greek) gives an algorithm for $GCD(x, y)$.
- Hilbert in 1900 challenges the math community to devise "a process according to which it can be determined by a finite number of operations" whether a polynomial has integral roots.
- They had examples of "algorithms", but no formal definition of what an algorithm was. To be able to prove that *no algorithm exists* for a computational problem, they needed a formal definition.
- 1936 Church and Turing each define a formal model of computation which later turned out to be equivalent.
- Halting and IntegerRoot problems turned out to have no algorithm.
- Computers were invented. What had been esoteric mathematics laid the theoretical ground work for a quick development of computer science.
- *Computational Complexity* began in the 60's when people like Jeff Edmonds' father developed the idea of *efficient algorithms* and poly time.

2.2 Computational Problems

Informally, a *Computational Problem* is a function from the set of possible inputs or instances to the set of possible outputs. For example, the input may be a multi-variate polynomial P like $x^3y + y^2z$ and the output may be a setting of the variables to integers that make the polynomial zero. An algorithm or machine is said to solve the problem if on every possible input, it eventually stops and gives the correct output. Most formal presentations of computational complexity, like Sipser's book, always assume that the inputs and the outputs are binary strings, $\{0, 1\}^n$. This allows them to have a standard input format and allows the bits of the string to be placed one per memory square. Formally:

Def^n: A *Computational Problem*, $F : \{0,1\}^* \Rightarrow \{0,1\}^*$ is function from binary strings to binary strings of finite length.

Sipser page 145 says "If we want to provide an object other than a string as input, we must first represent that object as a string. Strings can easily represent polynomials, graphs, grammars, automata, and any combination of those objects. A Turing machine may be programmed to decode the representation so that it can be interpreted in the way we intend. Our notation for the encoding of an object O into its representation as a string is $\langle O \rangle$. If we have several objects $O_1, O_2, \ldots, O_k$, we denote their encoding into a single string by $\langle O_1, O_2, \ldots, O_k \rangle$. The encoding itself can be done in many reasonable ways. It does not matter which one we pick." For example, our Integral Root problem would take the string $\langle P \rangle$ as input and output the string $\langle x, y, z \rangle$. Though it helps formally to encode everything into a string, to keep your intuition, one helpful thing to remember is the **type** of everything. For example, P is a multi-variate polynomial and x, y, and z are integers.

Def^n: The *size* of an object O is the number $| \langle O \rangle |$ of bits required to represent it wrt some fixed encoding scheme.

Most formal presentations of computational complexity, like Sipser's book, also focus on computational problems whose output is 1/0, yes/no, true/false. For example, we may simply ask, "Is there an integral root to the polynomial P" without requiring that the root be found. Then instead of representing a computational problem as a function $F : \{0,1\}^* \Rightarrow \{0,1\}$, we represent it as the set of inputs/instances for which the answer is yes, i.e. $L = \{x \mid F(x) = 1\}$.

Def^n: A *Language* L is a set of finite binary strings. An algorithm or machine is said to *decide* the language if on every finite binary string it eventually stops and gives the correct answer, i.e. *accepts* if the input is in L and *rejects* if it is not.

IMPORTANT: Remember that $x \in L$ means $F(x) = 1$.

Examples:

- $Co - Prime = \{\langle x, y \rangle \mid x \text{ and } y \text{ are integers and } GCD(x, y) = 1\}$.
- $IntegerRoot = \{\langle p \rangle \mid p \text{ is a multi-variate polynomial with an integral root } \}$.

2.3 Complexly Classes

We classify languages according to how difficulty they are to compute:

Def^n: A language L is said to be *decidable/computable* if there is a TM (algorithm/machine) such that on every finite binary string the TM eventually stops and gives the correct answer, i.e. *accepts* if the input is in L and *rejects* if it is not.

Def^n: A language L is said to be *undecidable/uncomputable* if it is not decidable. In other words, for every TM (algorithm/machine) there is at least one input on which either the machine runs for ever or the machine halts and gives the wrong answer.

There is a large class of interesting problems that are undecidable, but for which there is an algorithm that *verifies* the case when the answer is "yes".

Def^n: A language L is said to be *recognizable* if there is a TM (algorithm/machine) such that on every string in L, the TM eventually halts and accepts and on every string not in L it either halts and rejects or it runs for ever.

There is an asymmetry here between the "yes" and the "no" inputs. There are also languages whose "no" instances can be verified.

Def^n: A language L is said to be *co-recognizable* if the complement of L is recognizable. In other words, there is a TM (algorithm/machine) such that on every string in L, the TM either halts and accepts or it runs for ever and on every string not in L it eventually halts and rejects.

Def^n: A language L is said to be *polynomial time computable* (i.e. $L \in P$) if there is a TM (algorithm/machine) and a constant c such that on every binary string of length n the TM stops and gives the correct answer within at most n^c time steps.

There is a large class of important problems whose "yes" instances can be verified quickly with a little help from a goddess. In formally, a language L is said to in *Nondeterministic Polynomial Time* if the following is true. Suppose your boss give you an input I that happens to be in L (i.e. a "yes" instance) and he wants you to prove to him that the answer is "yes". You may not be able to do so in polynomial time on your own. However, again suppose that your personal deity helps you by giving you a witness w. Then in poly time with this witness, you can prove to your boss that $I \in L$. Your boss does not need to believe or even know about your deity. More over, if your deity turns out to be the devil, he cannot miss lead you, meaning that she cannot convince you that $I \in L$ when it is not. On the other hand, if your boss gives you an input I that has a "no" answer, then you cannot help him.

Def^n: A language L is said to be *nondeterministic polynomial time computable* (i.e. $L \in NP$) if there is a polynomially time computable verifier V such that for every input I, $I \in L$ if and only if there exists a poly sized witness w for which $V(I, w) = "Yes"$.

Again there is an asymmetry here between the "yes" and the "no" inputs. There are also languages whose "no" instances can be verified.

Def^n: A language L is said to be in *co-NP* if the complement of L is in NP.

2.4 Algorithms and Machines

Def^n: Given an input to a computational problem, an *algorithm* defines a process according to which the required output can be determined by a finite number of operations.

The word *machine* is used in the literature interchangeably with *algorithm*. In this context, "machine" is used to solve a specific problem as opposed to being a general purpose computer.

Example of an Algorithm:

The following is an example of an algorithm that repeats a bunch of bazaar steps and in the end claims to know the correct answer.

```
function GCD( a : int, b : int ) : int
    var x,y : int

    x := a
    y := b
    loop % LI: GCD(x,y) = GCD(a,b).
        exit when y=0
        xx := y
        yy := x mod y
        x := xx
        y := yy
     end loop
     result x
end GCD
```

On input $\langle a, b\rangle = \langle 22, 32\rangle$, the values of $\langle x, y\rangle$ will proceed as follows: $\langle 22, 32\rangle, \langle 32, 22\rangle, \langle 22, 10\rangle, \langle 10, 2\rangle, \langle 2, 0\rangle$ and the answer will be 2.

2.5 Formal Models of Computation

Def^n: A *Formal Model of Computation* defines what is a *legal* algorithm and what is not. Later when studying *Computational Complexity*, it will also define a cost measure to measure the cost of the computation. Such a cost might be the number of time steps or the number of memory cells used.

When defining a formal model of computation, one wants it to be general enough to allow every reasonable algorithm and restricted enough so as to not include unreasonable algorithms.

There are many different models of computation that turn out to be equivalent in terms of what computational problems they can compute. If fact, the Church/Turing thesis is that all *reasonable* models of computation are equivalent. This document considers the *CPU model*, the *C programming language model*, and the *Turing machine model*.

The reason for considering different models of computation are that they have different uses:

- With some models it is easier to envision and to express algorithms quickly and eloquently. Some programming languages like C, Java or Visual Basic make different programming tasks easier.
- Other models, like TMs, are as bare bones and as simple as you can imagine. You would never want to write an actual program in it. However, it is interesting that they are powerful enough to express every reasonable algorithm. They are useful when proving that no algorithm exists for a problem. Note, if there is not TM to solve a problem, then there is no C program that can solve it.

There are models of computation, like Deterministic Finite Automaton DFA and Push Down Automaton, that are less powerful. There are reasonable algorithms that are cannot be expressed in these models.

There are also models, like Circuits, Non-Deterministic Machines, and Quantum Machines that are more powerful. These models includes some unreasonable algorithms.

Before formally defining a model of computation, let us consider intuitively what properties a reasonable model of computation would have.

2.6 Properties of Models of Computation

1. *The algorithm/machine needs to be such that you can describe it completely in a constant amount of space, yet be able to solve every finite problem instances (input) no matter how large.*

   In the C model of computation, the algorithm is described in ASCII by the C program. In the CPU model, an algorithm is given in machine code. We will later see how to describe a Turing machine.

   Note, that both the algorithm and the input can have any arbitrarily large finite size. However, there is an order to this game: You must give me the algorithm before I give you the input. You could give me an algorithm that takes a trillion bits to write down, but then I could give you an input that takes a trillion trillion bits to write. Your algorithm would need to find a solution to my input in some finite amount of time.

   The following is an example of an algorithm that takes infinite space to write down:

```
if input = 1 return( 17 )
if input = 2 return( 24 )
if input = 3 return( 557 )
if input = 4 return( 142 )
if input = 5 return( 6 )
   ... for ever
```

   This type of algorithm could solve any computational problem because it explicitly stores all the answers. This is not interesting.

2. *The algorithm/machine has access to an unbounded amount of memory. Given a larger input, the machine is likely to use more of this memory. However, at any point in time it has only used a finite amount of it.*

   In the CPU model, this "unbounded" memory consists of the hard disk, and perhaps external tapes. In the C model, the program at run time can allocate as much memory as needed. In the Turing machine model, a one-way infinite tape of memory cells is provided. In each cell, a letter from some predefined finite alphabet can be written. You can think of this alphabet as being single bits, bytes, or ASCII. However, to be as general as possible, a TM is able to use any fixed alphabet it likes as long as it has a finite number of letters.

   In contrast, Deterministic Finite Automata, DFA, studied in the last course, did not have such memory. This limited what problems it could solve. Recall, for example, that a DFA cannot compute the computational problem $\{a^n, b^n \mid n \in \mathcal{N}\}$. An algorithm for this, needs at least $\log n$ bits of memory to count the a's.

3. *The input to the algorithm/machine can be of any arbitrary finite length. At the beginning of the execution, the bits of an n bit input are stored in the first n cells of the memory.*

4. *The algorithm/machine must proceed step by step in a deterministic fashion according to rules that are fully described by the algorithm. No leaps of magic are allowed.*

   Each time step, the algorithm/machine is in a *configuration*. Given the description of the machine and the description of the configuration, the configuration that the machine is in for the next step is determined easily. (This is why the algorithm is called deterministic).

   The configuration will specify, among other things, the complete contents of the memory. Hence, the configuration may be larger when the machine is given a larger input and may get larger through out time. However, because only a finite amount of memory has been at any point in time, the configuration can be written down using a finite number of bits.

   For example, in the GCD algorithm given above, the configuration would be described by $\langle X, Y \rangle$. On input $\langle 32, 22 \rangle$. The configuration after each loop are as follows: $\langle 22, 32 \rangle$, $\langle 32, 22 \rangle$, $\langle 22, 10 \rangle$, $\langle 10, 2 \rangle$, $\langle 2, 0 \rangle$.

5. *The model of computation defines a limited number of operations that can be performed. The input and the output of each operation must be of a bounded size.*

   Writing a letter in a specified cell of memory or multiplying two 32 bit numbers are reasonable operations.

   Given an arbitrary multi-variate polynomial and in one operation knowing whether it has integral roots is not a reasonable operation.

   Question: Is multiplying two arbitrarily large integers together in one step a reasonable operation? You might think ""yes"" because it is a natural well defined basic operation that you would want to do. However, someone else might think that finding integral roots is also a natural well defined basic operation. Who is to define what is reasonable and what is not? We will disallow both of these operations because their inputs and outputs are arbitrarily large. Also describing such an operation would require infinite space, which was disallowed in (1).

   For example, I know my 12*12 times tables off by heart. There are people who can instantly multiply 10 digit numbers together. But there is always some limit before some more involved algorithm is required.

   The C model of computation allows the addition and multiplication of 32 bit numbers but not larger. The CPU model may let you add but not multiply. The Turing machine model is the most generous. It allows the algorithm/machine to specify any operations it likes as long as the input, the output, and the description of the operation itself are all of bounded size. Recall the game. You can give me a TM that is able to multiply trillion bit numbers together in one step, and then I give you an input that contains a trillion trillion bits.

6. *The algorithm/machine has a central control that decides what operation to perform next. (Some models have more than one central control.) This central control has a bounded amount of knowledge about the overall configuration of the machine.*

   I heard of a psych study that stated that people can look at up to five objects and know "that is five", however, for larger numbers they need to group the objects and consider at most five of them at a time. The same idea happens here.

   Again recall the game. You say that your machine remembers a trillion bits of information and then I give you a trillion trillion bit input. On such an input, the machine may use a trillion trillion trillion cells of memory, but the local knowledge of the central control is always bounded by the fixed trillion.

   In the CPU model of computation, the central control "knows" the values in the main registers, and a program counter indicating the next machine instruction. It does not directly "know" the values of all of memory.

   In the C model, we could say that it "knows" the values of the local variables allocated during compile time, but not of the variables allocated during run time. It also knows the next line of C code to be executed.

   In the Turing machine model, a TM is defined with a finite set of states. During each time step, it is in one such state. What this central control "knows" is completely determined by the state that it is in.

   The central control cannot know/store the value of an arbitrary integer, because the integer may take up to much space. However, suppose what it knows is one integer between 0 and 9. Then it will have $|Q| = 10$ states, $q_0, \ldots q_9$. When it is in state q_3, it "knows" that the value is 3. Instead, suppose that it remembers two integers between 0 and 9. Then it does not have two sets of states. It has one set containing one state for every possible thing that its total knowledge might contain. In this case, it would have $|Q| = 10 \times 10 = 100$ states $q_{\langle 0,0\rangle}, q_{\langle 0,1\rangle}, \ldots, q_{\langle 9,9\rangle}$. If instead the central control remembers q bits of information, it must have $|Q| = 2^q$ states. If it knows two integers between 0 and 9 and q bits, it has $|Q| = 10 \times 10 \times 2^q$ states.

   When specifying a TM, one only needs to say the number of states (and later the transitions between them). It is only to help our personal understanding that we name one of the states $q_{3,4,cat}$ and interpret it to mean that $x = 3$, $y = 4$, and $s =$ "cat".

7. *Each time step, the central control is allowed to read and/or write to one memory cell of the unbounded memory. Most models of computation restrict how this is done in some way.*

   When the central control reads a memory cell, it learns more information. However, in doing so it must forget something, because at any given time step the total amount it knows is bounded.

   One key way that models of computation differ from each other is how the central control specifies which memory cell to read/write.

   In the CPU model, it stores the address of a memory cell in some register R and executes a machine code instruction like $ReadRA$ which reads into register A the value from the memory cell addressed by the value in register R.

   In the C model, the central control has pointer variables like p that "point" at the allocated data structures. The command $a = p->x$ stores in the central control's local memory the value in the x field of the data structure pointed to by p.

   One issue is that how the central control with a constant amount of space/knowledge/states is able to distinctly specify an unbounded number of memory cells.

   In the CPU model, if the R register only has 32 bits, then only $2^{32} = 4$ Giga different address can be referenced. Two such registers can reference 16 Giga Giga bytes. This is more then will ever be needed in practice, but still not unbounded. The CPU model get around this problem by saying "out of memory" if the problem ever comes up. In this way, it is not a theoretical model.

   In the C model, if the central control has a dozen local pointers, then it can only address a dozen different addresses at one time. The way it maintains an unbounded amount of data is by forming a

linked list. The way it accesses this data is by walking down the linked list, keeping in local memory a pointer to the next object in the list.

The Turing machine deals with this problem by not allowing the central control to access an arbitrary memory cell. A TM has a head which at each point in time is on one of the unbounded number of memory cells. The TM can only read/write to the cell that the head is currently on. As well, it cannot move the head arbitrarily, but one square to the left or to the right each time step. The TM does not know on which cell the head is located. Initially, the head is at the beginning of the tape and the TM could count the cells as it moves the head right. However, it will quickly run out of its own local memory to store the count. One option for the TM is to place markers in the cells of the tape. Then it can say move the head left until it finds the next such marker.

Another model of computation is called a *Random Access Machine* (RAM). It is the same as a Turing machine except that it is able to read/write from an arbitrary memory cell. This is achieved by allowing the central control to construct an address on a separate tape and then to access the addressed cell of the main memory tape in one step. Because this separate tape has an unbounded length, it can be used to address one of an unbounded number of different cells.

Given any of these models of computation, after T time steps, we know that the machine has addressed at most T of the memory cells. (It can only address one per time step). The other cells will still be blank. Hence, the configuration of the machine can include the contents of this unbounded number of memory cells by specifying only the contents of these T cells. The Turing Machine model has the additional advantage that because the head must move one cell at a time, we know that in T time steps it can access at most the first T contiguous cells.

8. *Each time step, the central control decides what to do, which operation to perform on its local information, what new information to store in its local information, what value to write to memory, and which cell of the memory to read and write to next. It bases this decision of what to do next solely on the information that it knows in its bounded knowledge and on the contents of the memory cell that it is currently reading. This is all that it knows.*

   The function between what the machine knows and what it will do next is specified by its *transition function*, which is denoted by δ.

   In the CPU and the C models of computation, the transition function is the machine code or the C code for the algorithm. Suppose that the 138^{th} line of code was $a = p- > x$. Recall that one of the things the machine "knows" is the current line number. So the transition function will say that if the machine knows that the current line is 138, then next thing it does is the operation $a = p- > x$. The machine also knows the value of p so the transition function says to fetch the value in the x field of the object pointed at.

   In the Turing machine model, all the central control knows is its current state q and the contents of the tape where the head is located. The only thing a TM is allowed to do is to write a character in the cell where the head is located, move the head left or right, and to specify a new current state. Hence, the transition function is a function $\delta : Q \times \Gamma \Rightarrow Q \times \Gamma \times \{L, R\}$, where Q is the set of states, Γ is the set of letters that can be written in a memory cell, and L and R indicate left and right. If the TM is in state q_i and is reading the character b and the transition function says $\delta(q_i, b) = (q_j, c, L)$, then the TM writes a c over the b in the cell where the head is located, moves the head one cell to the left and changes to state q_j.

   Recall that given the description of the machine, the machine configuration for the next step is determined easily from the configuration of the current configuration. For a TM, the configuration will specify the complete contents of the memory, the location of the head, and the current state. You can see that the transition function easily determines the next configuration from the current one.

   Though what a TM is allowed to do is seemingly very simple, it is in fact very powerful. As said, it can do any operation on its local knowledge. It does this by hardwiring in all the answers. For example, suppose we want our TM at some point to perform the operation $y = x * c \ mod \ 10$, where x and y are local variables and c is the value of the cell currently being read. Then we might give the TM a state

$q_{\langle 2,-,12,3,\ldots,8,y=x*c mod 10\rangle}$. For our own purposes, this state will mean, among other things, that $x = 2$, y has no value and the next operation to execute is $y = x * c \ mod\ 10$. We do this by defining the transition function to have $\delta(q_{\langle 2,-,12,3,\ldots,8,y=x*c \ mod\ 10\rangle}, 3) = (q_{\langle 2,6,12,3,\ldots,8,next\ op\rangle}, 3, R)$. Note that we changed the value of y to 6, and left the memory cell with the value 3.

9. *A final thing that an algorithm/machine is able to do is to stop and give the answer.*

   In the C model, this occurs when a return statement is reached in the main program.

   In the Turing machine, the TM has special accept and reject states. When it completes, it simply transitions to the accept state if it wants to claim the input is in the language, i,e. the answer is "yes" and transition to the reject state otherwise.

   If the TM wants to compute a function with a more general output, then in the end it can erase everything in memory except for the answer written in the first m cells.

This completes the informal properties that we want formal models of computation to have. You can now turn to section 3.1 of Sipser's book to read the formal definition of a TM.

Def^n: We say that one model of computation is at least as powerful as another if the first can simulate the second.

We can prove that TM = Multi Tape TM = RAM = CPU = C.

Chapter 3

DFAs, Iterative Programs, and Loop Invariants

DFAs, NFAs, and Regular Languages are all discussed very well in the Sipser book. Here we will discuss the connection between these and iterative programs and loop invariants.

3.1 Different Models and Notations for Simple Devices and Processes

This section studies simple algorithms, devices, processes, and patterns.

Examples:

- simple iterative algorithms
- simple mechanical or electronic devices like elevators and calculators
- simple processes like the job queue of an operating system
- simple patterns within strings of characters.

Similar Features: All of these have the following similar features.

Input Stream: They recieve a stream of information to which they must react. For example, the stream of input for a simple algorithm consists of the characters read from input; for a calculator, it is the sequence of buttons pushed; for the job queue, it is the stream of jobs arriving; and for the pattern within a string, one scans the string once from left to right.

Read Once Input: Once a token of the information has arrived it cannot be requested for again.

Bounded Memory: The algorithm/device/process/pattern has limited memory with which to remember the information that it has seen so far.

Models/Notations: Because they have these similar features, we are able to use the same *models of computation* or notations for describing them and thinking about them. They also help focus on the above listed aspects of the algorithm/device/process/pattern without being bogged down with details that are irrelevant to the current level of understanding.

There are, however, more than one such models/notations that can be used. Each provides different tools, insights, and levels of abstraction. One should know how to develop algorithm/devices/processes in each of these models and know how to mechanically compile an algorithm/device/process/pattern developed in any of these models into an equivalent one within any of the others. The models/notations considered are the following.

Deterministic Finite Automaton (DFA): A DFA, $M = \langle Q, \Sigma, \delta, s, F \rangle$, is an abstraction of a algorithm/device/process/pattern. It has a set of states $Q = \{q_1, q_2, \ldots, q_{|Q|}\}$ to represent the states that the device might be in and a transition function $\delta : Q \times \Sigma \to Q$ that dictates how the device responds to the next input token. If the DFA's current state is $q \in Q$ and the next input character is $c \in \Sigma$, then then next state of the DFA is given by $q' = \delta(q, c)$.

Though a DFA can be used to model any algorithm/device/process/pattern, this course will focus on using it as a model of computation. The input for the computation is fed character by character into the DFA. The computation on the DFA begins by initializing the DFA into the start state denoted by s. When the last character of the input has been read, the input is *accepted* if the DFA is in one of the accept states contained in the set denoted by F. If the last state is not an accept state, then the input is *rejected*.

A path through a graph: A DFA can be viewed as an edge labeled graph. The nodes of the graph correspond to the states of the DFA; the labeled edges of the graph to the transitions of the DFA; and paths through the graph correspond to computations on the DFA.

The graph representing the DFA has the property that for each node q and each character c of the alphabet Σ there is one edge labeled c coming out of node q. It follows that for every string $\omega \in \Sigma^*$ there is exactly one path through the graph, labeled with ω, starting at the start state (proof by induction on the length of the string). The DFA accepts a string ω if this path labeled ω ends in an accept state.

Two advantages of switching to this level of abstraction is that it is more visual and one then can draw on all the tools of graph theory.

An Iterative Program with a Loop Invariant: The above models sometimes seem esoteric to a beginning student. Hence, we will consider how such computations/devices/processes/patterns can be developed and expressed using standard code.

An algorithm that consists of small steps implemented by a main loop is referred to an *iterative algorithm*. Examples, will be given below. Formally, one proves that an iterative algorithm works correctly using what are called *loop invariants*. These same techniques can also be used to think about, develop and describe iterative algorithms so that they are easily understood.

We will see that the idea of a loop invariant also provides meaning to the states of a DFA. Instead of naming the states $q_0, q_1, q_2, \ldots$, it is better to give the states names that explain what the computation remembers about the part of the input read so far.

Nondeterministic Finite Automaton (NFA): Another very useful level of abstraction is *nondeterminism*. The idea is to write your algorithm assuming that you have some very powerful help from your favorite *higher power*. With this help, designing the algorithm is much easier. Once a correct algorithm has been designed within this level of abstraction, the next completely separate task is to *move between* the different levels of abstraction, by mechanically "compiling" your algorithm into an algorithm that does not require this help.

Nondeterministic Machine: The *transition function* of these machines does not provide only one next state that the machine must be in based on its current configuration. Instead it gives a set of possible states. The result is that each input instance does not have just one computation path, but has many. We say that the input instance is *accepted* if there exists at least one computation path that leads to an accept state.

Help From A Higher Power: One can view a computation on a nondeterministic machine as requiring nondeterministic help from a higher power. This power tells the computation the sequence of choices to make to lead it along one of the accepting computation paths to an accepting state. A key feature of this help is that a corrupt higher power can never convince you that there is an accepting computation when there is not one. The reason is that you are able to check that the sequence of choices given to you does in fact lead to an accept state.

Regular Languages: Regular expressions are a very useful notation for describing simple string structures. We will see how they also correspond to the computational problems that are solved by DFAs.

3.2 Iterative Programs and Loop Invariants

Structure: An iterative program (with bounded memory) has the following structure.

```
routine()
    allocate "state" variables
    initialize "state" variables
    loop
        % ⟨loop invariant⟩                What we remember about the input read so far
        exit when end of input
        allocate "temp" variables, eg. c
        get(c)                            % Reads next character of input
        code                              % Compute new values for state variables
        deallocate "temp" variables
    end loop
    if("state" variables have an "accept" setting) then
        accept
    else
        reject
    end if
end routine
```

Steps in Developing an Iterative Program: You may want to read my optional document on writing iterative programs using loop invariants that can be found on the web. The following are the main steps.

Assertions: Assertions are very helpful when thinking through the logic of the algorithm and when explaining the algorithm to someone else.

An assertion is made at some particular point during the execution of an algorithm. It is a statement about the current state of the data structure that is either true or false. If it is false, then something has gone wrong in the logic of the algorithm.

An assertion is not a task for the algorithm to perform. It is only a comment that is added for the benefit of the reader.

Some languages allow you to insert them as lines of code. If during execution such an assertion is false, then the program automatically stops with a useful error message. This is very helpful when debugging your program. However, it is often the case that the computation itself is unable to determine whether or not the assertion is true.

Loop Invariant: Suppose that you, as the computation, have read in some large prefix of the input string. With bounded memory you cannot remember everything you have read. However, you will never be able to go back and to read it again. Hence, you must remember a few key facts about the prefix read. The most difficult part of designing such an algorithm is knowing what to remember. The *loop invariant* is the assertion that the state variables store this key information about this prefix. The following are different types of information that should be remembered.

Whether to Accept This Prefix: The prefix of the input that you have read so far may be the entire input. You do not yet know whether the end has been reached. Therefore, the foremost thing that you must know about the current prefix is whether you would accept it if the input would stop here. Is this prefix itself in the language?

Progress: You will likely need to remember additional information about the prefix of the string read so far. The language/computational problem in question may have a number of conditions that must be met before it accepts an input string. Perhaps the prefix of the string read so far is not in the language because it has not met all of these requirements, however, it has met some of them. In such case, you must remember how much progress has been made.

On the other hand, the language may reject a string as soon as a number of conditions are met. In this case, you must remember how much progress has been made towards rejection.

Definitely Accept or Reject: Sometimes the prefix is such that no mater what the rest of the input string is, you will definitely accept (or reject) it. This would be good to remember.

Patterns: Sometimes the strings that are accepted contain a pattern that must be repeated over and over. In this case, you must remember at what point in this pattern this prefix ends.

Integrity Constraints: Sometimes certain settings of the state variables are not allowed. For example, your the amount in your bank account cannot be negative. The loop invariant would state these constraints.

State Variables: What you remember about the prefix of the input read so far is stored in the variables that we will refer to as the *state* variables. You must define what variables are needed to store this information. They can have only a fixed number of bits or be able to take on only fixed number of different values.

Maintaining Loop Invariant: You must provide code for the loop that maintains the loop invariant. Let ω denote the prefix of the input string read so far. You do not know all of ω, but by the loop invariant you know something about it. Now suppose you read another character c. What you have read now is ωc. What must you remember about ωc in order to meet the loop invariant? Is what you know about ω and c enough to know what you need to know about ωc?

You must write code to change the values of the state variables as needed. You may allocate as many temporary variables as needed to help you. For example, the variable c that reads in the next character of input is a temporary variable. These variables are deallocated at the bottom of the loop. Anything that you want to remember about ωc when back at the top of the loop must be stored in the state variables.

Initial Conditions: Now that we have an idea of where we are going, we can start at the beginning knowing which direction to head. At the beginning of the computation, no input characters have been read and hence the prefix that has been read so far is the empty string ϵ. Which values should the state variables have to establish the loop invariant?

When unsure, the initial conditions can be determined by working backwards. You know the loop invariant for when a single character has been read. Move backwards from there to see what conditions you want for the empty string.

Ending: When the input string has been completely read in, your loop invariant states that you know whether or not the string should be accepted. The steps outside the loop are to check whether the state variables have *accepting* or *rejecting* conditions and to accept or reject the string appropriately.

Testing: Try your algorithm by hand on a couple of examples. Consider both general input instances as well as special cases.

This completes the step for developing an iterative algorithm.

Example - Mod Counting: Consider the language

$$L = \{w \in \{0,1\}^* \mid \left[\ 2 \times (\#\text{ of 0's in } \omega) - (\#\text{ of 1's in } \omega)\ \right] \bmod 5 = 0\}$$

For example, with $\omega = 10011010011001001$, $2 \times (\#$ of 0's in $\omega) - (\#$ of 1's in $\omega) = 2 \times 9 - 8 = 10 \bmod 5 = 0$. Hence $\omega \in L$.

To construct an iterative program for this we follow the steps given above.

Loop Invariant: Suppose that you, as the computation, have read in some large prefix of the input string. The most obvious thing to remember about it would be the number of 0's and the number of 1's. However, with a large input these counts can grow arbitrarily large. Hence, with only bounded memory you cannot remember them. Luckily, the language is only concerned with these counts modulo 5, i.e. we count $1, 2, 3, 4, 0, 1, 2, 3, \ldots$ and $28 \bmod 5 = 3$. Even better we could remember $\left[\ 2 \times (\#$ of 0's in $\omega) - (\#$ of 1's in $\omega)\ \right] \bmod 5$.

State Variables: To remember this we will have one state variable denoted r that takes values from $\{0, 1, 2, 3, 4\}$. The loop invariant will be that for the prefix ω, $\lceil$ 2 $\times$ (# of 0's in ω) - (# of 1's in ω) $\rceil$ mod 5 = r.

Maintaining Loop Invariant: Knowing the count r for the current prefix ω and knowing new character c, we can compute what r will be for the new prefix ωc. If we read a 0, then the count increases by 2 mod 5. If we read a 1, then it decreases by 1 mod 5.

Initial Conditions: Initially, the number of 0's and 1's is both zero. $\lceil$ 2 $\times$ zero - zero $\rceil$ mod 5 = 0. Hence, we set r to 0.

Ending: The language accepts the string if the count r is 0.

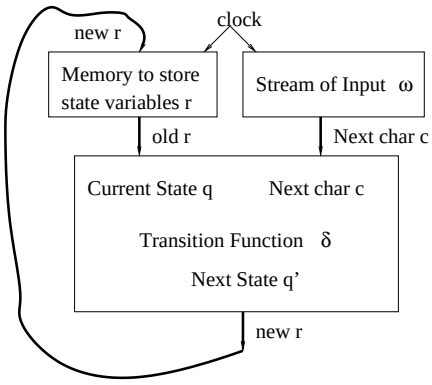
Code:

```
routine()
    allocate r ∈ {0,2,1,3,4}
    r = 0
    loop
        % ⟨loop invariant⟩        [2 × (# of 0's in ω) - (# of 1's in ω)] mod 5 = r
        exit when end of input
        allocate temp c
        get(c)                    % Reads next character of input
        if(c = 0) then            % Compute new values for state variables
            r = r + 2 mod 5
        else
            r = r − 1 mod 5
        end if
        deallocate c
    end loop
    if(r = 0) then
        accept
    else
        reject
    end if
end routine
```

We will give many more examples below.

3.3 Mechanically Compiling an Iterative Program into a DFA

Consider any iterative program P with bounded memory and an input stream that has already been developed. We could build a circuit with AND, OR, NOT gates that computes it. See Figure **??**.



We can also mechanically compiling this iterative program P into a DFA $M = \langle Q, \Sigma, \delta, s, F \rangle$ that solves the same problem. The steps are as follows.

Steps To Convert:

Alphabet Σ: The input alphabet Σ of the DFA M will contain one "character" for each possible token that the program P may input. This may be $\{0, 1\}$, $\{a, b\}$, $\{a, b, \ldots, z\}$, ASCII, or any other finite set of objects.

Set of States Q: When mapping a computation of program P to a computation of the DFA M, we only consider the computation of P at the points in which it is at the top of the loop. At these points in time, the *state* of the computation is given by the current values of all of P's state variables. Recall that the temporary variables have been deallocated. The DFA will have a set of states Q consisting of all such possible states that program P might be in.

The program's loop invariant asserts that its state variables are remembering some key information about the prefix of the input read so far. The DFA remembers this same information using its states. Instead of assigning the meaningless names $q_0, q_1, \ldots, q_{|Q|}$ to the states, more insight is provided both for developing and for understanding the device by having names that associate some *meaning* to the state. For example, the state named $q_{\langle even, first\ 0, last\ 1 \rangle}$ might remember that the number of 1's read so far is even, that the first character read was 0, and that the most recently read character was 1.

The set of states Q must contain a state for each such possible setting of the state variables. If the variables are allocated r bits of memory, then they can hold 2^r different values. Conversely, with $|Q|$ states a DFA can "remember" $\log_2 |Q|$ bits of information. No more.

Formally, suppose that the state variables in P are $v_1, v_2, \ldots, v_k$ and that the variable v_i can take on values from the set S_i. Then the complete set of states is

$$Q = S_1 \times S_2 \times \cdots \times S_k$$

$$= \left\{ q_{\langle v_1 = a_1, v_2 = a_2, \ldots, v_k = a_k \rangle} \mid a_1 \in S_1, a_2 \in S_2, \ldots, a_k \in S_k \right\}$$

For example, if the state variables remember a 6 bit string α, two digits $x, y \in \{0, 1, .., 9\}$, and a character $c \in \{a, b, c, .., g\}$ then the DFA would require $2^6 \times 10 \times 10 \times 7$ states, one of which might be named $q_{\langle \alpha = 100110, x=4, y=7, c=f \rangle}$.

Sometimes the set of states can be reduced. For example, any state that is not reachable by a path from the start state is clearly not useful. States that are "equivalent" in terms of future computation can be collapsed into one.

Transition Function δ: The DFA's transition function δ defines how the machine transitions from state to state. Formally, it is a function $\delta : Q \times \Sigma \to Q$. If the DFA's current state is $q \in Q$ and the next input character is $c \in \Sigma$, then the next state of the DFA is given by $q' = \delta(q, c)$.

We define δ as follows. Consider some state $q \in Q$ and some character $c \in \Sigma$. Set program P's state variables to the values corresponding to state q, assume the character read is c, and execute the code once around the loop. The new state $q' = \delta(q, c)$ of the DFA is defined to be the state corresponding to the values of P's state variables when the computation has reached the top of the loop again.

A great deal of complexity can be *hard wired* into the transition function δ of a DFA. For this reason, we can allow the code within P's loop to have any level of complexity. It can have if, select, or while statements. It can perform additions, multiplications, or any other operation. In fact, it can even perform uncomputable operations (like the $HALTING$ problem) as long as the input to the operation depends only on the current values the program's state and temporary variables.

For example, suppose that program P when reading the character $'f'$ factors a 100 digit number stored in its state variables. The DFA would need at least $2 \cdot 10^{100}$ different states. For each 100 digit number x, there are two states q_x and $q_{<x,a,b>}$ where $x = a \times b$ is a suitable factoring. Using

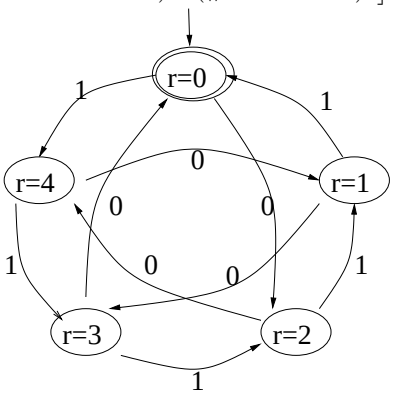
the transition function $\delta(q_x, f) = q_{<x,a,b>}$, the DFA can factor in one time step. In fact, we can simply assume that the DFA automatically knows any information that can be deduced from the information that it knows.

The Start State s: The start state s of the DFA M is the state in Q corresponding to the initial values that P assigns to its state variables.

Accept States F: A state in Q will be considered an *accept* state of the DFA M if it corresponds to a setting of the state variables for which P accepts.

This completes the construction of a DFA M from an iterative program P.

Example: Consider again the iterative program developed for the language $L = \{w \in \{0,1\}^* \mid \lceil\ 2 \times (\#\text{ of } 0\text{'s in } \omega) - (\#\text{ of } 1\text{'s in } \omega)\ \rceil \bmod 5 = 0\ \}$. Following these steps produces the following DFA.



3.4 More Examples

We will now present DFAs and regular expressions for the following languages. We recommend that you attempt to do them yourself before reading on.

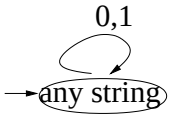
1. $L = \emptyset$.
2. $L = \{0,1\}^* - \{\epsilon\}$ = every string except ϵ.
3. $L = \{\omega \in \{0,1\}^* \mid \omega$ begins with a 1 and ends with a 0 $\}$
4. $L = 0^*1^*$.
5. $L = \{0^n1^n \mid n \geq 0\}$.
6. $L = \{w \in \{0,1\}^* \mid$ every 3^{rd} character of w is a 1 $\}$. Eg. $10\underline{1}00\underline{1}11\underline{1}01\underline{1}0 \in L$, $\epsilon \in L$, $0 \in L$, and $10\underline{0} \notin L$.
7. $L = \{\omega \in \{0,1\}^* \mid \omega$ has length at most three $\}$
8. $L_{AND} = \{\omega \in \{0,1\}^* \mid \omega$ has length at most three AND the number of 1's is odd $\}$
9. $L = \{\omega \in \{0,1\}^* \mid \omega$ contains the substring 0101 $\}$. For example $\omega = 1110101101 \in L$, because it contains the substring, namely $\omega = 111\ 0101\ 101$.
10. $L = \{J \in ASCII^* \mid J$ is a Java program that halts on input zero and $|J| \leq 10000\}$.
11. $L = \{\epsilon, 0, 010, 10, 11\}$.
12. Dividing an integer by seven.

13. Adding two integers.

When developing a DFA for a complex language it is useful to follow all the steps outlined above. For simple languages all the steps should be thought about, but many of them need not be written down. In the examples below, think for yourself about the steps that have been skipped.

Just Say No: $L = \emptyset$.

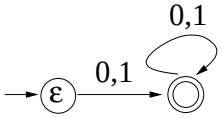
Nothing needs to be remembered about the prefix read so far because we simply reject every string.



(Zero information is remembered. The program uses zero bits for the state variables. The DFA needs $2^{zero} = 1$ states.)

Starting: $L = \{0,1\}^* - \{\epsilon\}$ = every string except ϵ.

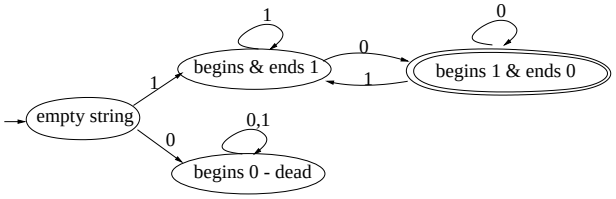
Here we must only remember whether or not we have seen any characters yet.



A regular expression representing L is $\Sigma\Sigma^*$.

More Info: $L = \{\omega \in \{0,1\}^* \mid \omega$ begins with a 1 and ends with a 0 $\}$

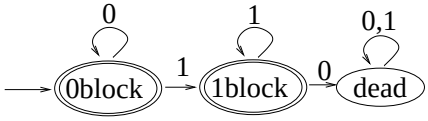
What the state variables must remember is which characters the prefix read so far begins and ends with (or that the prefix is empty and hence begins and ends with no characters). If the string begins with a 0, then there really is no reason to remember anything but this fact. In such a case, you enter what is sometimes referred to as a *dead* state. Once in such a state, the computation never leaves it and rejects the string when the computation is over.



A regular expression representing L is $1\Sigma^*0$.

Different Phases: $L = 0^*1^*$.

A string in this language contains a block of 0's followed by a block of 1's. You must remember which block or phase you are in. In the first phase you can read any number of 0's. When the first 1 is read, you must move to the second phase. Here you can read as many 1's as you like. If a 0 appears after the first 1 has been read, then the string is not in the language. Hence, the computation enters the dead state.



Note one should check special cases. For example, here either blocks could be empty, namely 000 and 111 are both valid strings. See how the DFA works for these cases.

Finite Memory: $L = \{0^n 1^n \mid n \geq 0\}$.

Phases: A string in this language also must contain a block of 0's followed by a block of 1's. Hence, you must remember the same things remembered in the last example.

Same Count: However, for this language, the two blocks must have the same number of characters. For example, $000111 \in L$ but $00011 \notin L$.

Suppose that a large block of 0's has been read in. What must you remember? Clearly, you need to know how many 0's have been read in. Otherwise, you will not know whether the number of 1's that follows is the same. Recall that you cannot go back and reexamine the block of 0's.

Code: An iterative program that works is as follows.

```
routine()
    allocate p ∈ {0block, 1block, dead}
    allocate n0, n1 : int
    p = 0block
    n0 = n1 = 0
    loop
        % ⟨loop invariant⟩                 p gives the phase, n0 the number of 0's,
                                            % and n1 the number of 1's
        exit when end of input
        allocate temp c
        get(c)                              % Reads next character of input
                                            % Compute new values for state variables
        if( p = 0block and c = 1 ) then
            p = 1block
        end if
        if( p = 1block and c = 0 ) then
            p = dead
        end if

        if(c = 0) then
            n0 = n0 + 1
        else
            n1 = n1 + 1
        end if

        deallocate c
    end loop
    if(p ≠ dead and n0 = n1) then
        accept
    else
        reject
    end if
end routine
```

Infinite Set of States Q: When converting this into a DFA, the first step is to construct the set of states Q consisting of all possible settings of the state variables. Here n_0 is allocated so that it can take on any integer. Hence, the corresponding set of state Q is infinite.

This iterative program can be converted into a very good deterministic automata. However, because it does not have a finite set of states, it is not a deterministic *finite* automata.

0^n1^n is Not Regular: We will later prove that no DFA accepts this language. The intuition is that in order to know whether or not the blocks have the same size, the computation must remember how many 0's have been read. An input can have an arbitrarily large number of 0's. Hence, a DFA, with only a bounded amount of memory, is unable to distinguish between all of these different counts.

When told this, students always protest. Here are the two common statements.

0*1*: Students give a DFA like the one above computing 0^*1^* and say that this DFA accepts the language 0^n1^n. I respond by saying that yes this DFA accepts all the strings that are in the language 0^n1^n, but it also accepts many strings like 00111 that are not in the language. For a DFA to *accept* a language, it must accept a string if and *only if* it is in the language.

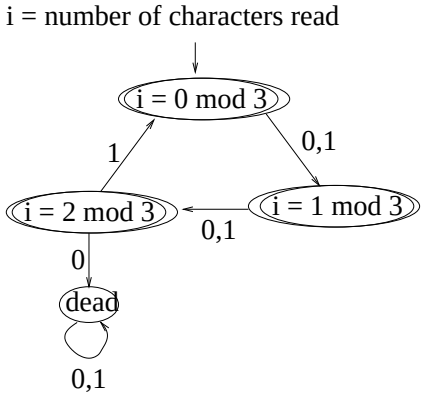
The Computation Game: Students also argue as follows. "Give me a string, say $0^{1000}1^{1000}$. I can construct a DFA with a finite number of states that determines whether or not the string is in the language."

This, however, is not the way that the computation game is played. If you are claiming that a language or a computational problem is computable, then you must first give me a machine/algorithm that you claim solves the problem. Then I, as an adversary, get to choose an input string for which your machine must work. If you give me a machine with 20 states, I could give you a string with 21 0's. However, if you give me a machine with a trillion states, I could give you a string with a trillion and one 0's. Your machine must still work. That is how the game is played.

Finite State Counting: A DFA is unable to count arbitrarily high. The next is a second example demonstrating how a DFA is able to count modulo some fixed constant. The subsequent example shows how a DFA is also able to count up to some bounded amount.

Mod Counting: $L = \{w \in \{0,1\}^* \mid \text{every } 3^{rd} \text{ character of } w \text{ is a } 1\}$. Eg. $10\underline{1}00\underline{1}11\underline{1}01\underline{1}0 \in L$, $\epsilon \in L$, $0 \in L$, and $10\underline{0} \notin L$.

The computation must remember modulo 3 how many characters have been read in, so that it knows which characters are every third one. It also must remember whether the string has already failed to meet the conditions.



A regular expression representing L is $(\Sigma\Sigma 1)^* (\epsilon \cup \Sigma \cup \Sigma\Sigma)$. Here the $(\Sigma\Sigma 1)^*$ ensures that every third character is a 1, but also forces the string's length to be a multiple of three. The $(\epsilon \cup \Sigma \cup \Sigma\Sigma)$ adds one zero, one, or two, more characters on the end so that the string can be any length.

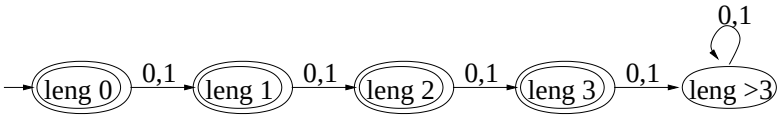
Counting to an Upper Bound: $L = \{\omega \in \{0,1\}^* \mid \omega \text{ has length at most three}\}$

The state variables must remember whether the length of the prefix read so far is 0, 1, 2, 3, or more than 3.

```
routine()
    ℓ = 0 and r = even
    loop
        % ⟨loop invariant⟩
            ℓ ∈ {0, 1, 2, 3, more} is the length of the prefix read.
            Whether the number of 1's in the prefex read is r ∈ {even, odd}
        exit when end of input
        get(c)      % Reads next character of input
        if( ℓ < 4) then ℓ = ℓ + 1
        if( c = 1) then r = r + 1 mod 2
```

end loop
if($\ell < 4$ AND r =odd) then
accept
else
reject
end if
end routine

A DFA would have the corresponding set of states and corresponding transitions.



A regular expression representing L is $\epsilon \cup \Sigma \cup \Sigma\Sigma \cup \Sigma\Sigma\Sigma$.

Intersection ≈ AND: $L_{AND} = \{\omega \in \{0,1\}^* \mid \omega$ has length at most three AND the number of 1's is odd $\}$

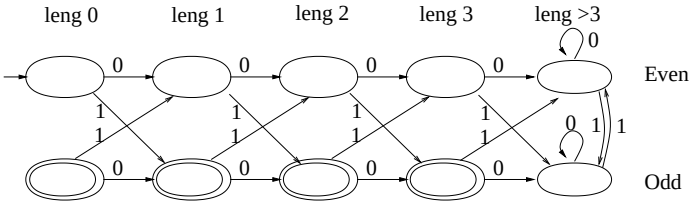
Membership within some languages, like this one, depends on more than one condition. This requires remembering more than one separate piece of information about the prefix. In an iterative program, this is done using separate state variables. The set of states of the DFA is constructed by taking the Cartesian product of these separate sets of values.

This iterative program will remember the length with the state variable $l \in \{0, 1, 2, 3, more\}$ and the parity of the number of 1's with $r \in \{even, odd\}$. The set of states of the DFA is the cross product of these sets of values, namely the $5 \times 2 = 10$ states $Q = \{0, 1, 2, 3, more\} \times \{even, odd\} = \{q_{\langle 0,even\rangle}, q_{\langle 0,odd\rangle}, q_{\langle 1,even\rangle}, q_{\langle 1,odd\rangle}, q_{\langle 2,even\rangle}, q_{\langle 2,odd\rangle}, q_{\langle 3,even\rangle}, q_{\langle 3,odd\rangle}, q_{\langle more,even\rangle}, q_{\langle more,odd\rangle}\}$.

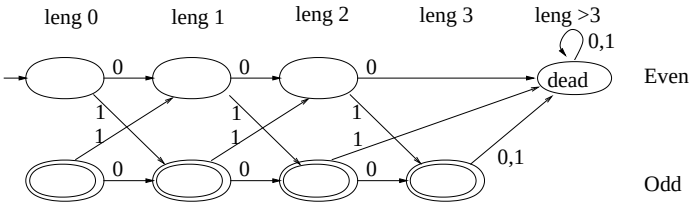
Within the loop, the program has separate code for updating l and r, namely
if($l \neq more$) then $l = l + 1$
$r = r + c \bmod 2$

The transition function δ for the DFA changes the state mirroring how these two separate pieces of code change the values of the state variables, namely $\delta(q_{\langle l,r\rangle}, c) = q_{\langle l+1}$ unless it is already more,$_{r=r+c \bmod 2\rangle}$.

Initially, the program sets $l = 0$ and $r = even$. In the end, the program accepts if $l \neq more$ AND $r = odd$. The start state s and the set of accept states F reflect these settings.



Or more simply



How would the DFA change for the language $L_{OR} = \{\omega \in \{0,1\}^* \mid \omega$ has length at most three OR the number of 1's is odd $\}$?

A regular expression representing the language of strings with an odd number of 1's is $0^*10^*\left(10^*10^*\right)^*$, representing L_{AND} is $\{1, 01, 10, 100, 010, 001, 111\}$, and representing L_{OR} is $(\epsilon \cup \Sigma \cup \Sigma\Sigma \cup \Sigma\Sigma\Sigma) \cup \left(0^*10^*\left(10^*10^*\right)^*\right)$.

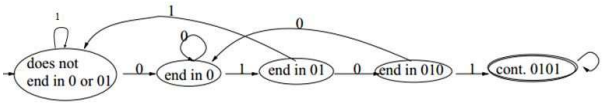
Partial Progress: $L = \{\omega \in \{0,1\}^* \mid \omega \text{ contains the substring } 0101\ \}$. For example $\omega = 1110101101 \in L$, because it contains the substring, namely $\omega = 111\ 0101\ 101$.

If the prefix of the string read so far contains the substring 0101, this certainly should be remembered. However, if it does not, some *progress* may have been made towards this goal. A prefix grows one character to the right at a time. Therefore, if the prefix ends in 010 a lot of progress would have been made because reading a 1 would complete the 0101 task. The levels of progress are

1. $q_{\langle \text{contains } 0101 \rangle}$
2. $q_{\langle \text{ends in } 010\text{, but does not contain } 0101 \rangle}$
3. $q_{\langle \text{ends in } 01\text{, but does not contain } 0101 \rangle}$
4. $q_{\langle \text{ends in } 0\text{, but does not end in } 010 \text{ and does not contain } 0101 \rangle}$
5. $q_{\langle \text{does not ends in } 0 \text{ or in } 01 \text{ and does not contain } 0101 \rangle}$

Note how we were careful to ensure that these classifications are disjoint and cover all strings so that every possible prefix is contained in exactly one of these. The loop invariant asserts that when we have read some prefix, we are in the correct corresponding state.

When defining the code for the loop or the transition function for the DFA, you must determine when a character moves you forward to a higher level of progress and when you are moved backwards. As said, if the prefix ω so far ends in 010 and you read the character $c = 1$, then progress is made because the prefix $\omega 1$ ends in and hence contains 0101. On the other hand, if ω ends in 010 and you read a $c = 0$, then this destroys the chance that the ending 010 will lead to a 0101. However, it is not a complete disaster. The prefix $\omega 0$ now ends in 0100. This last 0 begins again the pattern 0101. Care needs to be taken to determine all of these transitions.



A regular expression representing L is $\Sigma^* 0101 \Sigma^*$.

Finite Language: Every language that contains only a finite number of strings can be computed by a DFA. The intuition is that every finite language has a longest string length l_{max}. Hence, a DFA, which can remember any bounded amount of information, can either reject an input for being to long or remember it in its entirety.

Consider the example,

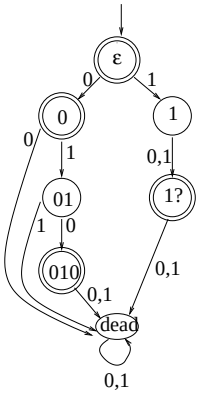
$$L = \{J \in ASCII^* \mid J \text{ is a Java program that halts on input zero and } |J| \leq 10000\}$$

The iterative algorithm simply reads in the input J and remembers the first 10000 characters. At the end of the computation, if the string is shorter then this, then the DFA, knowing the entire input, can magically know whether or not to accept it. If the string is longer, the DFA can simply reject it.

The DFA is constructed as follows. Its graph is a balanced tree of state nodes each representing a different input string.

$$\begin{aligned}
&Q = \{q_{\langle J \rangle} \mid J \in ASCII^* \text{ and } |J| \leq 10000\} \cup \{q_{\langle tooLong \rangle}\} \\
&s = q_{\langle \epsilon \rangle} \\
&\delta(q_{\langle J \rangle}, c) = q_{\langle Jc \rangle}, \text{ if } |J| < 10000 \\
&\qquad\qquad = q_{\langle tooLong \rangle}, \text{ otherwise} \\
&\delta(q_{\langle tooLong \rangle}, c) = q_{\langle tooLong \rangle} \\
&F = \{q_{\langle J \rangle} \mid J \in ASCII^*,\ J \text{ is a Java program that halts on input zero, and } |J| \leq 10000\}
\end{aligned}$$

Eliminating States: As said, languages containing a finite number of strings can be accepted by a DFA whose graph is a complete balanced tree. However, for some finite languages some of these states can be eliminated, because they contain the same information. Consider for example the language $L = \{\epsilon, 0, 010, 10, 11\}$. The following DFA accepts it.



Dividing: Dividing an integer by seven is reasonably complex algorithm. It would be surprising if it could be done by a DFA. However, it can. Consider the language $L = \{w \in \{0, 1, .., 9\}^* \mid w$ is divisible by 7 when viewed as an integer in normal decimal notation $\}$.

Consider the standard algorithm to divide an integer by 7. Try it yourself on say 39462. You consider the digits one at a time. After considering the prefix 394, you have determined that 7 divides into 394, 56 times with a remainder of 2. You likely have written the 56 above the 394 and the 2 at the bottom of the computation.

The next step is to "bring down" the next digit, which in this case is the 6. The new question is how 7 divides into 3946. You determine this by placing the 6 to the right of the 2, turning the 2 into a 26. Then you divide 7 into the 26, learning that it goes in 3 times with a remainder of 5. Hence, you write the 3 next to the 56 making it a 563 and write the 3 on the bottom. From this we can conclude that 7 divides into 3946, 563 times with a remainder of 5.

We do not care about how many times 7 divides into our number, but only whether or not it divides evenly. Hence, we remember only the remainder 2. One can also use the notation $395 = 2 \bmod 7$. To compute 3956 mod 7, we observe that $3956 = 395 \cdot 10 + 6$. Hence, $3956 \bmod 7 = (395 \cdot 10 + 6) \bmod 7 = (395 \bmod 7) \cdot 10 + 6 \bmod 7 = (2) \cdot 10 + 6 \bmod 7 = 26 \bmod 7 = 3$.

More formally, the algorithm is as follows. Suppose that we have read in the prefix ω. We store a value $r \in \{0, 1, .., 6\}$ and maintain the loop invariant that $\omega = r \bmod 7$, when the string ω is viewed as an integer. Now suppose that the next character is $c \in \{0, 1, ..., 9\}$. The current string is then ωc. We must compute $\omega c \bmod 7$ and set r to this new remainder in order to maintain the loop invariant. The integer ωc is $(\omega \cdot 10 + c)$. Hence, we can compute $r = \omega c \bmod 7 = (\omega \cdot 10 + c) \bmod 7 = (\omega \bmod 7) \cdot 10 + c \bmod 7 = r \cdot 10 + c \bmod 7$. The code for the loop is simply $r = r \cdot 10 + c \bmod 7$.

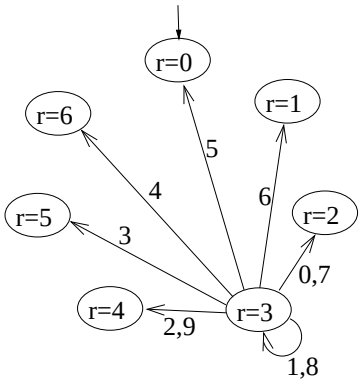
Initially, the prefix read so far is the empty string. The empty string as an integer is 0. Hence, the initial setting is $r = 0$. In the end, we accept the string if when viewed as an integer it is divisible by 7. This is true when $r = 0$. This completes the development of the iterative program.

The DFA to compute this will have seven states $q_0, \ldots, q_6$. The transition function is $\delta(q_r, c) = q_{\langle r \cdot 10 + c \bmod 7 \rangle}$. The start state is $s = q_0$. The set of accept states is $F = \{q_0\}$.

```
DivisibleBy7()
    allocate r ∈ [0..6].
    loop
        % ⟨loop invariant⟩: When viewing the string α read so far
        %                   as an integer x,
        %                   r is the remainder when dividing x by 7.
```

```
        exit when end of input
        get(c)    % Reads next character of input
        % r_new = "α_new" mod 7
        %       = "α_old c" mod 7
        %       = x_old × 10 + c mod 7
        %       = (x_old mod 7) ×10 + c mod 7
        %       = r_old × 10 + c mod 7
        r = r × 10 + c mod 7
    end loop
    if(r = 0) then
        accept
    else
        reject
    end if
end routine
```

$r_{new} = r_{old} * 10 + 4 \bmod 7$
$= 3 * 10 + 4 \bmod 7$
$= 34 \bmod 7$
$= 6$



Adding: The input to the problem consists of two integers x and y represented as strings of digits. The output is the sum z also represented as a string of digits. The standard algorithm is a classic iterative algorithm that depends on a loop invariant. As a reminder, add the following numbers.

```
  1896453
+ 7288764
----------
```

The input can be view as a stream if the algorithm is first given the lowest digits of x and of y, then the second lowest, and so on. The algorithm outputs the characters of z as it proceeds. The only memory required is a single bit to store the carry bit. Because of these features, the algorithm can be modeled as a DFA.

Alphabets Σ and Σ_{out}: A single token inputted in the Adding example, consists of a pair of digits, $\langle x_i, y_i \rangle$, one from x and one from y. The alphabet is

$$\Sigma = \{0..9\} \times \{0..9\} = \{\langle x_i, y_i \rangle \mid x_i, y_i \in \{0..9\}\}$$

The input $x = 312,\ y = 459$ is given as the string of characters
$\langle x_1, y_1 \rangle \langle x_2, y_2 \rangle \ldots \langle x_n, y_n \rangle = \langle 2, 9 \rangle \langle 1, 5 \rangle \langle 3, 4 \rangle$.

The algorithm will output the digits z_i of the sum $z = x + y$. They will be outputted from the low to the high order digits. Hence, output alphabet Σ_{out} is $\{0, 1, .., 9\}$.

Code:

```
routine()
    allocate carry ∈ {0, 1}
    carry = 0
    loop
        % ⟨loop invariant⟩  The sum of the characters read has been calculated and outputted.
                            % The state variable carry contains the final carry.
        exit when end of input
        allocate temp variables x_i, y_i, z_i, and s
        get(⟨x_i, y_i⟩)
        s = x_i + y_i + carry
        z_i = low order digit of s
        carry = high order digit of s
        put(z_i)
        deallocate temp variables x_i, y_i, z_i, and s
    end loop
    if(carry = 1) then
        put(carry)
    end if
end routine
```

DFA with Output: One can also construct a DFA that produces a string of output each time step. To do this the transition function is changed to $\delta : Q \times \Sigma \to Q \times \Sigma^*_{out}$, where Σ_{out} is the output alphabet. If the DFA's current state is $q \in Q$ and the next input character is $c \in \Sigma$, then the DFA outputs the string α and its next state is q', where $\delta(q, c) = \langle q, \alpha \rangle$. In a graph representation of such a DFA, the edge $\langle q, q' \rangle$ would be labeled with $\langle c, \alpha \rangle$.

The DFA:

$Q = \{q_{\langle carry=0 \rangle}, q_{\langle carry=1 \rangle}\}$.
$s = q_{\langle carry=0 \rangle}$
$\delta(q_{\langle carry=c \rangle}, \langle x_i, y_i \rangle) = \langle q_{carry=c'}, z_i \rangle$
where c' is is the high order digit and z_i is the low order digit of of $x_i + y_i + c$.

We do not need accept states, because the problem does not require the machine to accept or reject the input.

Calculator: Invariants can be used to understand a computer system that, instead of simply computing one function, continues dynamically to take in inputs and produce outputs. See Chapter **??** for further discussion of such systems.

In a simple calculator, the keys are limited to $\Sigma = \{0, 1, 2, \ldots, 9, +, clr\}$. You can enter a number. As you do so it appears on the screen. The + key adds the number on the screen to the accumulated sum and displays the sum on the screen. The *clr* key resets both the screen and the accumulator to zero. The machine only can store positive integers from zero to 99999999. Additions are done mod 10^8.

algorithm *Calculator*()

⟨*pre−cond*⟩: A stream of commands are entered.

⟨*post−cond*⟩: The results are displayed on a screen.

```
begin
    allocate accum, current ∈ {0..10^8 − 1}
    allocate screen ∈ {showA, showC}
    accum = current = 0
    screen = showC
    loop
```

$\langle loop-invariant \rangle$: The bounded memory of the machine remembers the current value of the accumulator and the current value being entered. It also has a boolean variable which indicates whether the screen should display the current or the accumulator value.

get(c)
if($c \in \{0..9\}$) then
    $current = 10 \times current + c \bmod 10^8$
    $screen = showC$
else if($c =' +'$) then
    $accum = accum + current \bmod 10^8$
    $current = 0$
    $screen = showA$
else if($c =' clr'$) then
    $accum = 0$
    $current = 0$
    $screen = showC$
end if
if($screen = showC$) then
    display($current$)
else
    display($accum$)
end if
end loop
end algorithm

The input is the stream keys that the user presses. It uses only bounded memory to store the eight digits of the accumulator and of the current value and the extra bit. Because of these features, the algorithm can be modeled as a DFA.

Set of States: $Q = \{q_{\langle acc,cur,scr \rangle} \mid acc, cur \in \{0..10^8 - 1\}$ and $scr \in \{showA, showC\}\}$. Note that there are $10^8 \times 10^8 \times 2$ states in this set so you would not want to draw the diagram.

Alphabet: $\Sigma = \{0, 1, 2, \ldots, 9, +, clr\}$.

Start state: $s = q_{\langle 0,0,showC \rangle}$.

Transition Function:

- For $c \in \{0..9\}$, $\delta(q_{\langle acc,cur,scr \rangle}, c) = q_{\langle acc,10 \times cur+c,showC \rangle}$.
- $\delta(q_{\langle acc,cur,scr \rangle}, +) = q_{\langle acc+cur,0,showA \rangle}$.
- $\delta(q_{\langle acc,cur,scr \rangle}, clr) = q_{\langle 0,0,showC \rangle}$.

Syntax Preprocessor: Your task here is to define a finite state transducer that will read strings representing C programs and write onto the output tape the same string/program with all comments removed. Comments in a C program start with /* and continue until the next */. For example,

```
input:
   if(x=5) /* This is a "great" comment */ then /* add 1 */ x=x+1
output:
   if(x=5) then x=x+1

input:
   x=5; /* Nested comments /* do not */ work like you might want. */ y=7;
output:
   x=5; work like you might want. */ y=7;

input:
```

```
   x=5; /* Comments that do not end should still be deleted.
output:
   x=5;
```

Comments cannot begin within a string literal, i.e. inside a quoted piece of text like "the /* here is not the start of a quote."

```
input:
   printf("This is not a /* comment */ ok"); /* This is */
output (of your machine, not of printf):
   printf("This is not a /* comment */ ok");
```

Do not worry about the syntax of the *C* program. Just treat it as a string that may have some occurrences of /* ... */ inside it. To really simplify things, assume that the only symbols in the input string are the following:

```
/ (slash)
* (star)
" (open quote or close quote - use same symbol)
a,b (to represent other characters)
```

The *C* complier has a preprocessor option that strips away comments. You can try it using

cc -E file

The Iterative Algorithm: Having the computation remember the last character read makes the code easier, but then there would be more states. My algorithm remembers only whether the last character is special or not.

routine()
    allocate $mode \in \{none, comment, quote\}$
    allocate $lastchar \in \{notspecial, special\}$
    $mode = none$
    $lastchar = notspecial$
    loop
        % ⟨loop invariant⟩
            % What the computation remembers about the prefix of the string read in so far is
            % whether at this point in the string it is with in a comment or quote,
            % and whether the last character was a "special".
            % If you are in neither a comment nor a quote,
            % then '/' is "special", because it might start '/*'.
            % If you are in a comment, then '*' is "special", because it might start '*/'.
            % If you are in a quote, then there are no special characters.
        exit when end of input
        allocate temp variables c and $thischar$
        get(c)

        % Determine whether this character c is special
        if(($mode = none$ and $c =' /'$) or ($mode = comment$ and $c =' *'$)) then
            $thischar = special$
        else
            $thischar = notspecial$
        endif

% Check if comment before outputting
if($mode = none$ and $lastchar = special$ and $c =' *'$) $mode = comment$

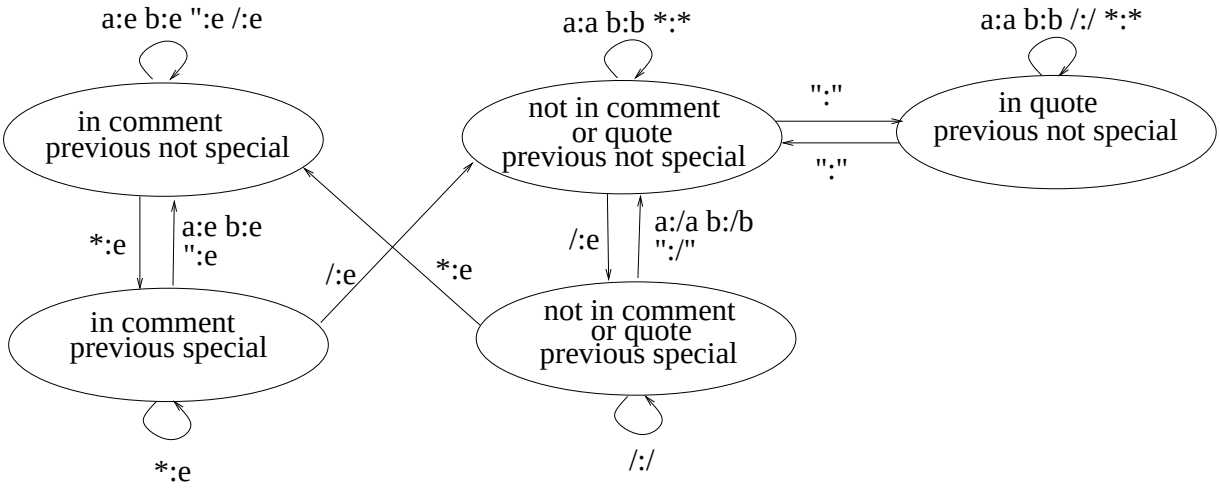
% Output characters. Note "special" characters always get printed next step.
if($mode \neq comment$) then
    if($lastchar = special$) output('/')
    if($thischar \neq special$) output(c)
endif

% Change mode
if($mode = comment$ and $lastchar = special$ and $c =' /'$) $mode = none$
if($mode = quote$ and $c =' "'$) $mode = none$
if($mode = none$ and $c =' "'$) $mode = quote$

$lastchar = thischar$
deallocate temp variables c and $thischar$
end loop

% Output last special character.
if($mode \neq comment$ and $lastchar = special$ output('/')
end routine

A DFA with Output: Recall that a DFA that produces an output has a transition function $\delta : Q \times \Sigma \rightarrow Q \times \Sigma^*_{out}$.



Harder Problem: Above we implied that comments that begin with a /* yet do not contain an ending */ should be deleted. Suppose that we change the description of the problem so that such comments are NOT deleted.

```
input:
   x=5; /* Comments that do not end should not be deleted.
output:
   x=5; /* Comments that do not end should not be deleted.
```

If possible construct a DFA for this new problem. If it is not possible give intuition as to why.

There is NO DFA to solve this problem. The reason is that you must remember the comment so that you can print it out if the line ends without a closing */. You cannot remember this with only a finite memory.

Chapter 4

DFA and Regular Complexity Classes

4.1 Closure

Given a DFA for L_1 and one for L_1 we can construct on for $L_1 \cap L_2$, for $L_1 \cup L_2$, and for $\overline{L_1}$.

routine$_1$()
    $\ell = 0$

    loop
        % $\langle$loop invariant$\rangle$
            $\ell \in \{0, 1, 2, 3, more\}$ is the length
            of the prefix read.

        exit when end of input
        get(c)   % Reads next character of input

        if($\ell < 4$) then $\ell = \ell + 1$
    end loop

    if($\ell < 4$) then
        accept
    else
        reject
    end if
end routine

routine$_2$()
    r = even

    loop
        % ⟨loop invariant⟩
            Whether the number of 1's in the
            prefix read is $r \in \{even, odd\}$.

        exit when end of input
        get(c)   % Reads next character of input

        if($c = 1$) then $r = r + 1 \bmod 2$
    end loop

    if(r =odd) then
        accept
    else
        reject
    end if
end routine

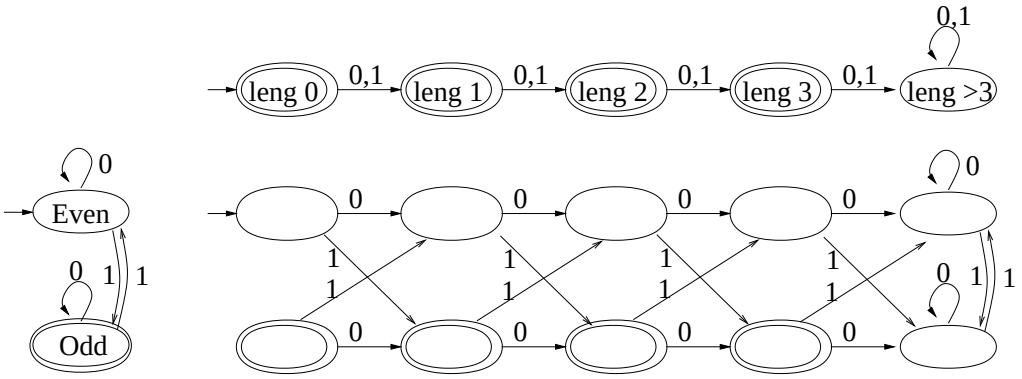
routine$_\cap$()
    $\ell = 0$ AND r = even

    loop
        % ⟨loop invariant⟩
            $\ell \in \{0, 1, 2, 3, more\}$ is the length of the prefix read. AND Whether the number of 1's in the prefix read is $r \in \{even, odd\}$.

        exit when end of input
        get(c)   % Reads next character of input

        if($\ell < 4$) then $\ell = \ell + 1$ AND if($c = 1$) then $r = r + 1 \bmod 2$
    end loop

    if($\ell < 4$ AND r =odd) then
        accept
    else
        reject
    end if
end routine



4.2 Converting NFA to DFA using the Clones and the Iterative Program Levels of Abstraction

Clones Level of Abstraction: A computation on an NFA can be viewed as a parallel computation in which many sub-processors are computing simultaneously. We will refer to these sub-processors as *clones* because they are all identical and because one is able to split into many. Given an NFA M_{NFA} and an input string α, the clones compute as follows.

Starting: Initially a single clone starts on the start state s_{NFA} of M_{NFA}.

Operations: Cones are able to repeatedly do either of following:

- Read the next input character and follow an edge labeled with the character read or
- Follow an edge labeled ϵ.

Splitting: Whenever a clone has a choice it splits into one clone for each option.

- If current state of a clone has two edges leaving it labeled ϵ, then the clone will split into three copies. One will read the next character, one will follow the one ϵ edge, and one will follow the other ϵ edge.
- Suppose a clone reads the next input character and it is a c. If his current state has five edges leaving it labeled c, then the clone will split into five copies, one for each edge.

Separate Inputs: Because these clones are acting independently, they need different copies of the input. When a clone reads the *next* input, he gets the next character that he has not seen.

Accepting The Input: We say that this computation of the NFA M_{NFA} on input α accepts if and only if at least one of the clones has read the entire input and is on an accept state.

Synchronizing the Clones: In asynchronous parallel computations, the sub-processors, having their own clocks, run at their own speeds. Sometimes, if they want to share information, they must synchronize. This involves some sub-processors waiting for others to catch up.

NFA to DFA: A DFA can only read the next character of the input once. If we are to simulate an NFA computation on a DFA, then all the clones must read the next character of the input all at the same time. If two clones follow a different number of ϵ edges, then they are not ready to read the next character at the same time. Hence, they must synchronize before reading it.

Super-Step: We will call the time from one synchronization to the next a *super-step*. For $i \geq 1$, the i^{th} super-step consists of the following sub-steps in the following order:

- All the clones simultaneously read the i^{th} input character.
- If the character read is a c, then each clone must first follow one edge labeled c.
- Each clone can then follow as many edges labeled ϵ as it likes, i.e. zero or more.

ϵ Moves First: One may ask why a clone within a super-step is not able to follow ϵ edges before reading the next character. The answer is that one could define a super-step to allow this, however, we have chosen not to. This restriction does not restrict what the clones are able to do for the following reason. Suppose within one super-step a clone reads the input character c and follows a few ϵ edges. Suppose that in the next super we allow it to follows a few more ϵ edges before reading the next input character. We can keep the sequence of sub-steps the same, but redefine the super-steps to have all of these ϵ moves within the first super-step. This ensure that the second super-step begins with reading an input character.

Before the First Super-Step: At the beginning of the computation, a clone may follow a number of ϵ edges from the start state before reading the first input character. Because of this, we allow a zeroth super-step within which each clone can follow as many ϵ edges as it likes.

At Most One Clone per State: As stated, a clone splits whenever it has a choice. The following are ways of getting rid of excess clones.

Dying: A clone dies if it reads a character for which there are no edges leaving the clone's current state.

Merging: Suppose that two clones are on the same state at the end of a super-step. Note they have read the same prefix ω of the input. Even if their pasts are completely different, everything that they remember about their past and about the prefix ω is characterized by which state of the NFA they are on. Being in the same state, what they remember is the same. Assuming that they have free will, what they choose to do in the future may be different. However, their potential futures are the same. Hence, at the moment, these clones are indistinguishable. Because of this, we will merge them into one clone. (If given a choice, they will split again.) The advantage of this is that at the beginning of each super-step each state of the NFA will either have zero or one clone.

Super-State: At any given point in time during a computation on a parallel machine, each sub-processor is in some internal state. The *super-state* of the entire parallel machine is determined by which state each of its sub-processors in in.

Similarly during a computation on an NFA, each clone is in one of the NFA states. The state of the entire NFA computation is determined by which sub-state each clone is in. There is nothing distinguishing between the clones. Hence, we do not need to say that clone Joe is in NFA state q_{38}. We only need to know how many clones are in each of the NFA's states. Because clones in the same NFA state merge into one clone, each NFA state contains either zero or one clone. Therefore, we can describe the super-state of the NFA computation by specifying for each NFA state, whether or not it contains a clone. An equivalent way of doing this is by specifying the set of NFA states $S = \{q \mid \text{a clone is on state } q\} \subseteq Q_{NFA}$. The set of possible super-states is the set of all subsets of Q_{NFA}, which is denoted either $Power(Q_{NFA})$ or $2^{Q_{NFA}}$. The second notation refers to the fact that there are $2^{|Q_{NFA}|}$ different subsets of Q_{NFA}.

NFA to Iterative Program: We can use a simple iterative program to simulate and to better understand the computation of the clones on the NFA. The steps in constructing any iterative program are the following.

1. We must decide what data structures or "state" variables are needed.
2. The loop invariant states what is remembered about the prefix ω of the input read so far. We must decide what will be remembered and how the "state" variables store this information.
3. We must figure out how this loop invariant is to be maintained as the next input character is read.
4. We must figure out how this loop invariant is initially obtained before the first input character is read.
5. When the entire input has be read in, we must determine based solely on the contents of the "state" variables whether or not to accept the input.

We will now consider each of these steps.

"State" Variables: The state variables of our iterative program must store the current super-state of the NFA. This consists of a subset $S \subseteq Q_{NFA}$ of the NFA states.

The Loop Invariant: For $i \geq 0$, after i times around the loop, the state variable S is storing the set $\{q \mid \text{a clone is on state } q \text{ after the } i^{th} \text{ super-step, i.e. before reading the } i+1^{st} \text{ input character } \}$ $= \{q \mid \text{there is a path from the start state of the NFA to state } q \text{ labeled } \omega, \text{ where } \omega \text{ is the prefix consists of the first } i \text{ characters of the input } \}$.

Maintaining the Loop Invariant: Suppose that at the beginning of the i^{th} super-step, S_{old} specifies which NFA states contain clones. Suppose that the next character read is a c. We must determine the locations S_{new} of the clones after this super-step. We know that the clones must follow an edge labeled c followed by any number of ϵ moves. Hence S_{new} is computed as follows.

$\delta(q, c)$: A clone on NFA state q after reading the input character c will split into clones which will move onto the NFA states in $\delta(q, c) = \{q' \mid$ the NFA has an edge from state q to state q' labeled $c\}$.

$\delta(S_{old}, c)$: After reading the c, the clones on states in S_{old} will split and move to the states in $\delta(S_{old}, c) = \cup_{q \in S_{old}} \delta(q, c) = \{q' \mid$ the NFA has an edge from some state $q \in S_{old}$ to state q' labeled $c\}$.

Closure: A set is said to be closed under an operation if for any object (objects) the result of the operation is still in the set. For example, the set of integers is closed under addition.
The closure of a set R under an operation is defined to be the smallest set $E(R)$ that contains the original R and is closed under the operation. For example, the closure $E(R)$ of the set $R = \{1\}$ under additions is the set of positive integers. 1 must be in the closure because it is in R. Because 1 is in the closure, $1 + 1 = 2$ must be in it. Because 1 and 2 are in the closure, $1 + 2 = 3$ must be in it, and so on. There is no reason for 3.5 or -4 to be in the closure. Hence, they are not.

$E(R)$: Given a set $R \subseteq Q_{NFA}$ of NFA states, define the set $E(R)$ to be those states reachable from R by zero or more ϵ moves, namely $E(R) = \{q'' \mid$ there is a path of any number of ϵ edges from q' to q'' where $q' \in R\}$.

$S_{new} = E(\delta(S_{old}, c))$: S_{new} consists of the NFA states that can be reached from S_{old} with one c edge followed by any number of ϵ edges.

The Starting Super-State: The loop invariant states that after 0 times around the loop, the state variable S is storing the set $\{q \mid$ a clone is on state q after the 0^{th} super-step, i.e. before reading the 1^{st} input character $\}$. Before reading this first input character, the clones can travel from the NFA's start state s_{NFA} along any number of ϵ edges. Hence, the set of NFA states that the clones might be in is $E(s_{NFA})$. Setting $S = E(s_{NFA})$ makes the loop invariant initially true.

Finishing: When the entire input has been read, the loop invariant states that the state variable S is storing the set $\{q \mid$ a clone is on state q after the last super-step $\}$. We are to accept the input if there is a clone on an accept state of the NFA. Hence, we accept if S is such that $S \cap F_{NFA} \neq \emptyset$.

The Iterative Program: Combining these pieces gives the following iterative program.

```
routine()
    allocate "state" variable S
    S = E(s_NFA)
    loop
        % ⟨loop invariant⟩          Clones are on the NFA states specified in S
        exit when end of input
        get(c)                      % Reads next character of input
        S = E(δ(S, c))
    end loop
    if(S ∩ F_NFA ≠ ∅) then
        accept
    else
        reject
    end if
end routine
```

NFA to DFA: To construct a DFA from an NFA one could convert the NFA into the iterative program and then convert the iterative program into a DFA as we have learned. Alternatively, one could construct the DFA directly.

Formal Translation: Given an NFA $M_{NFA} = \langle Q_{NFA}, \Sigma, \delta_{NFA}, s_{NFA}, F_{NFA} \rangle$ construct a DFA $M_{DFA} = \langle Q_{DFA}, \Sigma, \delta_{DFA}, s_{DFA}, F_{DFA} \rangle$ as follows.

- $Q_{DFA} = powerset(Q_{NFA}) = 2^{Q_{NFA}} = \{q_S \mid S \subseteq Q_{NFA}\}$.

- $\delta_{DFA}(q_{S_{old}}, c) = q_{S_{new}}$, where $S_{new} = E(\delta(S_{old}, c))$ consists of the NFA states that can be reached from S_{old} with one c edge followed by any number of ϵ edges.
- $s_{DFA} = q_{S_{start}}$, where $S_{start} = E(s_{NFA})$ consists of the NFA states that can be reached from the NFA's start state s_{NFA} with any number of ϵ edges.
- $F_{DFA} = \{q_S \mid S \cap F_{NFA} \neq \emptyset\}$ consist of those super-states that have clones on one of the NFA's accepts states.

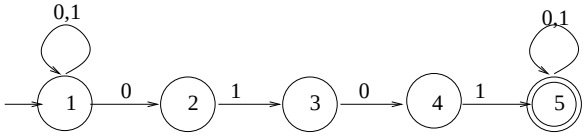
A Exponential Blow up in Size: Note that if the NFA has $|Q_{NFA}|$ states, then the DFA constructed this way has $2^{|Q_{NFA}|}$ states. Some times much smaller DFA can be constructed. Some times the smallest DFA is really exponentially bigger than the smallest NFA for the same language.

Constructing the DFA in a Practical Way: If you are given a picture of an NFA and you must quickly draw a picture of an equivalent DFA, the above definition of the DFA is not that useful. Instead follow the following practical steps.

1. The first step is to consider one at a time each tuple $\langle q, c\rangle$, where $q \in Q_{NFA}$ is a state of the NFA and $c \in \Sigma$ is a character. Assume that at the beginning of a super-step there is only one clone and he is one state q and that he reads the character c. Determine the set $E(\delta(q, c))$ of states that clones will be on at the end of the super-step. Recall that the clone must take a c move, possibly followed by any number of ϵ moves. Put all of these results into a table.
2. The set of states of the DFA is stated to be $Q_{DFA} = 2^{Q_{NFA}}$, however, it may not need all these states. A primary reason for not needing a state q_S is because it is not reachable from the start state. The following technique considers only DFA states that are reachable from the DFA's start state.
   Start by writing down the start state $s_{DFA} = E(s_{NFA})$ of M_{DFA}. Repeat the following until every state of M_{DFA} has one edge leaving it for each character of Σ.
   - Choose an a super-state q_S of your M_{DFA} and a character c for which your M_{DFA} does not have a c transition.
   - Put a clone on the NFA states in the set S corresponding to DFA state q_S. Have them take a super-step, i.e. make a c followed by as many ϵ transitions as they like.
   - Determine where the clones will be as follows. $S_{new} = E(\delta(S, c)) = \cup_{q \in S} E(\delta(q, c))$. You can read each $E(\delta(q, c))$ from your table.
   - If your DFA does not already contain the super-state $q_{S_{new}}$, then add it.
   - Add the transition edge from q_S to $q_{S_{new}}$ labeled c.
3. Determine which states of your M_{DFA} are accept states. Remember q_S is an accept state if S contains an NFA accept state.
4. The M_{DFA} that you constructed may still be too big. Is there any obvious way of simplifying M_{DFA}? Why does M_{DFA} still work with this simplification?

Recall what was said about the closure of a set under an operation. The above technique is forming the closure under the δ operation starting with the start symbol s_{DFA}.

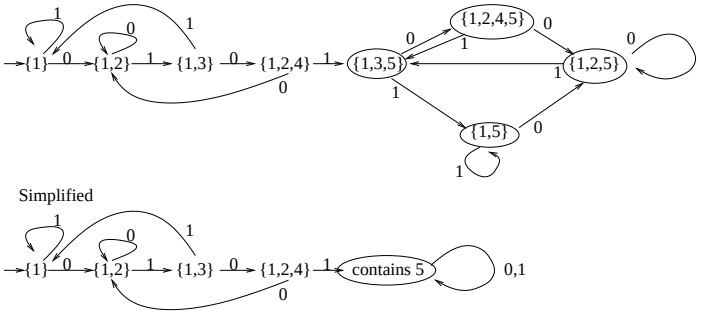
An Example: Consider the NFA M_{NFA}:



1. What language does it accept?
   - Answer: $L = \{w \in \{0,1\}^* \mid w$ contains the substring 0101 $\}$.
2. Fill in the table of $E(\delta(q, c))$.
   - Answer:

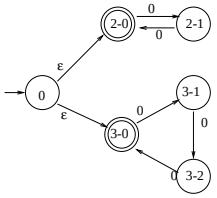
$Q\backslash\Sigma$	0	1
1	$\{1,2\}$	$\{1\}$
2	$\emptyset$	$\{3\}$
3	$\{4\}$	$\emptyset$
4	$\emptyset$	$\{5\}$
5	$\{5\}$	$\{5\}$

3. Construct the DFA M_{DFA} for the states that are reachable. Is there any obvious way of simplifying it?
   - Answer:



4. Does this DFA look familiar?
   - Answer: This DFA is in the notes.

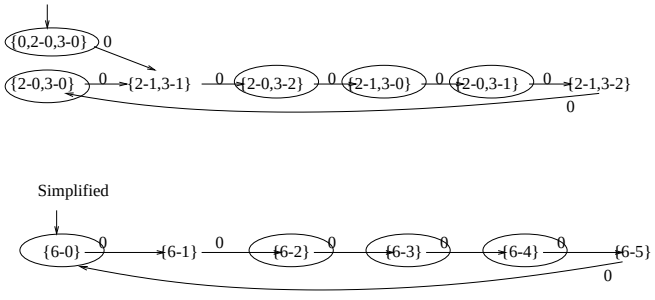
Another Example: Consider the NFA M_{NFA}



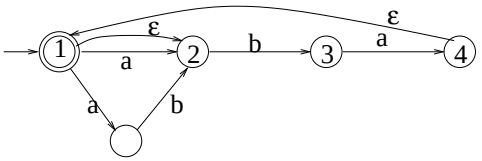
1. What language does it accept?
   - Answer: $L(m) = \{w \in \{0\}^* \mid |w| = 0 \ mod \ 2 \ or \ |w| = 0 \ mod \ 3\}$
2. Fill in the table of $E(\delta(q, c))$.
   - Answer:

$Q\backslash\Sigma$	0
0	$\emptyset$
$2-0$	$\{2-1\}$
$2-1$	$\{2-0\}$
$3-0$	$\{3-1\}$
$3-1$	$\{3-2\}$
$3-2$	$\{3-0\}$

3. Construct the DFA M_{DFA} for the states that are reachable. Is there any obvious way of simplifying it?
   - Answer:



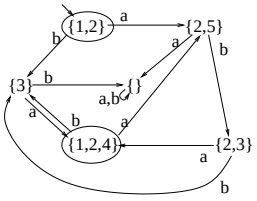
4. Make a DFA M'' for the following language $L'' = \{w \in \{0\}^* \mid |w| \text{ is 0, 2, 3, or 4 mod 6}\}$.
   - Answer: See above fig.
5. Is there a connection between M' and M''? Why? (If you are in the mood, see references for the Chinese Remainder Theorem of number theory.)
   - Answer: They compute the same language. The Chinese Remainder Theorem tells you for example that if $x = 1 \ mod \ 2$ and $x = 2 \ mod \ 3$ then $x = 5 \ mod \ 6$.



Another Example: Consider the NFA M_{NFA}

1. What language does it accept?
   - Answer: $((\epsilon \cup a \cup ab)ba)^*$ or $(ba \cup aba \cup abba)^*$
2. Construct an equivalent DFA M_{DFA}.
   - Answer:

	a	b
1	{2,5}	{}
2	{}	{3}
3	{1,2,4}	{}
4	{}	{}
5	{}	{2}



4.3 Overview of Results

For each of the following theorems, give a two or three sentence sketch of how the proof goes or why it is not true.

1. Every DFA M can be converted into an equivalent NFA M' for which $L(M) = L(M')$.
   - Answer: True. A DFA M is automatically also a NFA.
2. Every NFA M can be converted into an equivalent DFA M' for which $L(M) = L(M')$.
   - Answer: True. M' computing is like having many clones computing on M. The current state of M' is the subset of the states of M that the clones are currently on.
3. Every DFA M can be converted into an equivalent TM M' for which $L(M) = L(M')$.
   - Answer: True. A TM M' can simulate a DFA M simply by not using its tape except for reading the input in.
4. Every TM M can be converted into an equivalent DFA M' for which $L(M) = L(M')$.
   - Answer: False. A TM can compute $L = \{0^n 1^n \mid n \geq 0\}$ and a DFA cannot.
5. Every regular expression R can be converted into an equivalent NFA M for which $L(R) = L(M)$.
   - Answer: True. NFA are closed under union, concatenation, and kleene star. Hence, the NFA M is built up by following these operations within the R.

6. Every DFA M can be converted into an equivalent regular expression R for which $L(M) = L(R)$.

   - Answer: True. This is the complex algorithm in which states of the NFA are removed one at a time and edges are allowed to be labeled with regular expressions.

7. Every NFA M can be converted into an equivalent one M' that has a single accept state.

   - Answer: True. Given M, construct M' by adding a new state f that will be the only accept state of M'. Add an ϵ transition from each accept state of M to f.

8. The set of languages computed by DFA is closed under complement.

   - Answer: True. Given a DFA M that computes L, construct M' by switching the accept and not accept states of M. $L(M') = \overline{L(M)}$.

9. The set of languages computed by NFA is closed under complement.

   - Answer: True. Given a NFA M that computes L, constructing M' for which $L(M') = \overline{L(M)}$ cant be done directly. Instead, convert the NFA M into a DFA M'' as done above and then construct from the DFA M'' the DFA M' by switching the accept state, so that $L(M') = \overline{L(M'')} = \overline{L(M)}$.

4.4 Pumping vs Bumping Lemma

The goal here is to present a technique for proving that a language is not decidable by a DFA. The method found in every text and course I have seen is the *Pumping Lemma*. I think that the students find this technique hard, esoteric, useless, ... and I tend to agree with them. Here is an alternative that I think is more intuitive, easier to prove, and easier to use. Let me know if anyone has any reason that it is not as good as the pumping lemma. Let me know if you have a language that is not regular and this technique either does not prove or does not prove as easily as the pumping lemma.

Pumping Lemma: For every language L, the pumping lemma (as given in class) defines a game G_L between a prover and an adversary. Then the lemma states "If L is regular, then the adversary has a strategy for the game G_L in which he is guaranteed to win". The contra-positive of this statement is "If there are no winning strategies for the adversary to the game G_L, then L is not regular". If a game has a winning strategy for the prover than it cannot have a winning one for the adversary. It follows that one can prove that a language L is not regular simply by presenting a strategy for the prover in which he is guaranteed to win.

Bumping Lemma: Consider a language L that we want to prove is not regular. We say that the prefixes α and β can be *distinguished* by L if there exists a postfix γ such that $\alpha\gamma$ and $\beta\gamma$ are given different answers by L. Eg. if L_{names} is the set of names of professors at York, then the first names $\alpha = jeff$ and $\beta = gordon$ are distinguished by L because the last name $\gamma = edmonds$ is such that such that $\alpha\gamma = jeffedmonds$ is in L_{names} and $\beta\gamma = gordonedmonds$ is not. Similarly, if $L_{0^n1^n} = \{0^n1^n \mid n \geq 0\}$, the prefixes $\alpha = 0^i$ and $\beta = 0^j$ for $i \neq j$ are distinguished by L because the postfix $\gamma = 1^i$ is such that such that $\alpha\gamma = 0^i1^i$ is in $L_{0^n1^n}$ and $\beta\gamma = 0^j1^i$ is not.

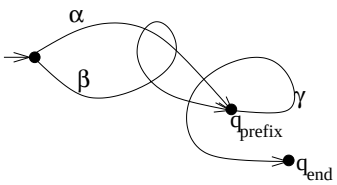
We say that the set $S \subseteq \Sigma^*$ of prefixes can be *distinguished* by L if every pair of prefixes $\alpha, \beta \in S$ can be distinguished. Likely the set S_{names} of first names of professors at York can be distinguished by L_{names}. We have proved that $S = 0^*$ can be distinguished by $L_{0^n1^n}$.

Theorem 1 *If language L distinguishes the prefixes S, then any DFA that decides L requires at least $|S|$ states.*

The intuition is that after the DFA has read in some prefix in S, it needs enough memory to remember which one it has seen or equivalently it needs enough states to distinguish between them.

Corollary 2 *If language L distinguishes the prefixes S, where S contains an infinite number of prefixes, then L is not a regular language, because any DFA deciding it requires an infinite number of states.*

Proof of Theorem 1:



By way of contradiction, assume that the DFA M decides L and has fewer than $|S|$ states. For each prefix $\alpha \in S$, run M on it and place it on the state of M at which the machine ends up. The *pigeon hole principle* states that you can't put $n+1$ pigeons into n pigeon holes without putting more than one pigeon in the same hole. By this principle, there is a state of M which we will denote q_{prefix} that receives two prefixes $\alpha, \beta \in S$. Because the prefixes α and β can be distinguished by L, there exists a postfix γ such that $\alpha\gamma$ and $\beta\gamma$ are given different answers by L. I claim that the state q_{end} the M ends on when run on $\alpha\gamma$ is the same as when run on $\beta\gamma$. The reason is that after reading α and β, M is in the same state q_{prefix} and then it continues in the same way from there when reading γ. This last state q_{end} is either an accept state or is not. Hence, it cannot give the correct answer on both input instances $\alpha\gamma$ and $\beta\gamma$, which have different answers. ■

```
Hi Jeff,

I finally got a chance to look at the handout you sent me a while
ago (Bumping the Pumping Lemma).  I really think it is much easier
to use than the Pumping Lemma, and it captures my intuition about
why languages aren't regular much better.  How did students like it?

I believe I can prove a converse to your Corollary 2:

Claim:  If there is no infinite set distinguished by L, then L is regular.

This means that your "Bumping Lemma" can be used (in principle) to
prove that any non-regular language is non-regular (provided you can
find the right set).  I suspect that this lemma might be part of the
general folklore in automata theory.  Nevertheless, it might actually
be worth it to publicise the Bumping Lemma as a pedagogical tool,
since all the automata textbooks seem to give only the Pumping Lemma.
(In view of the claim above, it's actually better because it's an "iff"
characterization of regular languages.)  Maybe a little note in the
Education Forum of SIGACT News?

Proof of claim:  Let S be a maximal distinguished set.  Construct a
DFA whose states are labelled by elements of S.

For any x in S and a in Sigma:
If xa is in S, then delta(x,a)=xa.
Otherwise, there is a y in S s.t. L does not distinguish xa,y
(if there were no such y, S union {xa} would be a larger
distinguished set).  Let delta(x,a)=y.

If epsilon is in S, use epsilon as the starting state s_0.  Otherwise,
there is some string y in S that is indistinguishable from epsilon.
```

```
Use that as the starting state s_0.

The accepting states of the DFA are those strings of S that are in L.

Lemma:  For any string w, L does not distinguish w from
delta*(s_0, w).

Proof:  (It's actually pretty trivial, but I'll jot it down formally
just to be careful.)

Base case (w = empty string) follows from defn of s_0.
Suppose lemma holds for w.  Let a \in \Sigma.  Prove it for wa.

Case I:  delta*(s_0,w)a \in S.  Then delta*(s_0,wa)=delta*(s_0,w)a.
So if some postfix z could be used to distinguish wa from
delta*(s_0,wa), then the postfix az could be used to distinguish w from
delta*(s_0,w), contrary to ind. hyp.

Case II:  delta*(s_0,w)a \notin S.  Then delta*(s_0,wa)=y, where y is
some string in S that L does not distinguish from delta*(s_0,w)a.
If there is a postfix z s.t. L would give different answers on waz and
yz, then (since L gives same answer on yz and delta*(s_0,w)az) L would
give different answers on waz and delta*(s_0,w)az.  This means that
the postfix az could be used to distinguish w from delta*(s_0,w),
contrary to the ind. hyp.

It follows from the lemma and the defn of accepting states that the
DFA decides the language L.

Eric
```

For each of the following languages, attempt on your own to prove whether is regular or not. For contrast, we provide for each a proof using both this technique and using the pumping lemma.

1. $L = \{a^n b^m \mid n \geq m\}$
2. $L = \{a^n b^m \mid n \geq 100, m \leq 100\}$
3. $\{a^n \mid n = k^2 \text{ for some } k \geq 0\}$
4. $L = \{a^n b a^m b a^{n+m} \mid n, m \geq 0\}$
5. $L = \{ww \mid w \in \{a, b\}^*\}$
6. $L = \{uww \mid u, w \in \{a, b\}^*\}$
7. $L = \{w \in \{0, 1\}^* \mid w \text{ when viewed as an integer in binary is divisible by 7 }\}$.

1. $L = \{a^n b^m \mid n \geq m\}$

   - Answer: This L is not regular because it distinguishes between the infinite number of prefixes in the set $S = a^*$. The prefixes $\alpha = a^i$ and $\beta = a^j$ for $i < j$ are distinguished by L as follows. Let $\gamma = b^j$. Then $\alpha\gamma = a^i b^j$ is not in L and $\beta\gamma = a^j b^j$ is.

- Answer: It is not regular. The proof is the following strategy to win the game G_L.
  - Adversary gives pumping number p.
  - I give $s = a^p b^p$. Note $s \in L$ and $|s| \geq p$.
  - Adversary gives split $s = xyz$, where $|xy| \leq p$ and $|y| > 0$. Due to the fact that $|xy| \leq p$ and s begins with p as, it follows that y must contain all as.
  - I set $i = 0$.
  - $xy^iz = xz = a^{p-|y|}b^p$. This is not in L because $n = p - |y| < p = m$.

2. $L = \{a^n b^m \mid n \geq 100, m \leq 100\}$

- Answer: It is regular. $\{a^n \mid n \geq 100\}$ and $\{b^m \mid m \leq 100\}$ are both clearly regular. L, which is their concatenation, is also regular.

3. $L = \{a^n \mid n = k^2 \text{ for some } k \geq 0\}$

- Answer: This L is not regular because it distinguishes between the infinite number of prefixes in the set $S = a^*$. The prefixes $\alpha = a^i$ and $\beta = a^j$ for $i < j$ are distinguished by L as follows. Let $\gamma = a^r$ for some r where $i + r$ is a perfect square and $j + r$ is not. Then $\alpha\gamma = a^{i+r}$ is in L and $\beta\gamma = a^{j+r}$ is not. There are lots of r that work. One is $r = j^2 - i$. Trivially $i + r = j^2$ is a perfect square. We show that $j + r$ is not a perfect square because it lies strictly between two consecutive perfect squares, namely $j + r = j^2 + (j - i) > j^2$ because $i < j$ and $j + r = j^2 + (j - i) < j^2 + 2j + 1 = (j + 1)^2$.

- Answer: It is not regular. Claim for every integer p there is a pair of consecutive squares p_0 and p_1 that are further apart than p. For example, let p_0 be p^2 and p_1 be $(p + 1)^2$. Then $p_1 - p_0 = (p + 1)^2 - p^2 = 2p + 1 > p$.

  The proof that L is not regular is the following strategy to win the game G_L.
  - Adversary gives pumping number p.
  - I give $s = a^{p^2}$. Note $s \in L$ and $|s| \geq p$.
  - Adversary gives split $s = xyz$, $|xy| \leq p$ and $|y| > 0$. Clearly y contains only as.
  - I set $i = 2$.
  - xy^2z has length $p^2 + |y|$. Note $p^2 < |xy^2z| = p^2 + |y| \leq p^2 + p < (p + 1)^2$. Therefore, this length $p^2 + |y|$ is not a square and hence the string xy^2z is not in L.

4. $L = \{a^n ba^m ba^{n+m} \mid n, m \geq 0\}$

- Answer: This L is not regular because it distinguishes between the infinite number of prefixes in the set $S = a^*$. The prefixes $\alpha = a^i$ and $\beta = a^j$ are distinguished by L as follows. Let $\gamma = baba^{i+1}$, then $\alpha\gamma = a^i baba^{i+1}$ is in L and $\beta\gamma = a^j baba^{i+1}$ is not.

- Answer: It is not regular. The proof is the following strategy to win the game G_L.
  - Adversary gives pumping number p.
  - I give $s = a^p ba^p ba^{2p}$. Note $s \in L$ and $|s| \geq p$.
  - Adversary gives split $s = xyz$, where $|xy| \leq p$ and $|y| > 0$. Due to the fact that $|xy| \leq p$ and and s begins with p as, it follows that y must contain all as from the first block.
  - I set $i = 0$.
  - $xy^iz = xz = a^{p-|y|}ba^p ba^{2p}$. This is not in L because $(p - |y|) + p \neq 2p$.

5. $L = \{ww \mid w \in \{a, b\}^*\}$

- Answer: This L is not regular, but the obvious proof is buggy. L distinguishes between the infinite number of prefixes in the set $S = \{a, b\}^*$. The prefixes $\alpha = w$ and $\beta = w'$ are distinguished by L with $\gamma = w$ because $\alpha\gamma = ww$ is in L and $\beta\gamma = ww'$ is not. This is buggy because if $w = 00$ and $w' = 0000$ then $ww' = 000\ 000$ is in L. The same proof works by changing S to ab^* so that with $\alpha = \gamma = w = ab^i$ and $\beta = w' = ab^j$, $\beta\gamma = ww' = ab^iab^j$ is not in L.

- Answer: It is not regular. The proof is the following strategy to win the game G_L.
  - Adversary gives pumping number p.
  - I give $s = ab^pab^p$. Note $s \in L$ and $|s| \geq p$.
  - Adversary gives split $s = xyz$, where $|xy| \leq p$ and $|y| > 0$. Due to the fact that $|xy| \leq p$ and that the second a is at index $p + 2$, it follows that y does not contain the second a or the bs after it.
  - I set $i = 0$.
  - $xy^iz = xz$ either only has one a or is of the form $ab^{p-|y|}ab^p$ which is not in L.

6. $L = \{uww \mid u, w \in \{a, b\}^*\}$

   - Answer: Trick question. This language IS regular. I claim that L is in fact the language $\{a, b\}^*$ containing all strings. Why? Consider any string $u \in \{a, b\}^*$. This is in L simply by setting w to ϵ.

7. $L = \{w \in \{0, 1\}^* \mid w$ when viewed as an integer in binary is divisible by 7 $\}$.

   - Answer: This language IS regular. We did it in the notes.

Chapter 5

Context Free Grammars

Context Free Grammars consists of set of rules and a start *non-terminal* symbol. Each rule specifies one way of replacing a non-terminal symbol in the current string with a string of terminal and non-terminal symbols. When the resulting string consists only of terminal symbols, we stop. We say that any such resulting string has been *generated* by the grammar.

Context free grammars are used to understand both the syntax and the semantics of many very useful languages like mathematical expressions, JAVA, and English.

The *Syntax* of a language involves which strings of tokens are valid sentences in the language. We will say that a string is in the language if it can be generated by the grammar.

The *Semantics* of a language involves the "meaning" associated with strings. In order for a compiler or natural language recognizer to determine what a string means, it must *parse* the string. This involves deriving the string from the grammar and in doing so determining which parts of the string are "noun phrases", "verb phrases", "expressions", or "terms".

Most first algorithmic attempts to parse a string from a context free grammar requires $2^{\Theta(n)}$ time. However, there is an elegant dynamic programming algorithm (we may consider it in this course) that parses a string from any context free grammar in $\Theta(n^3)$ time. Though this is impressive, it is much too slow to be practical for compilers and natural language recognizers.

Some context free grammars have a property called *look ahead one*. Strings from such grammars can be parsed in linear time by what I consider to be one of the most amazing and magical recursive algorithms.

This recursive algorithm very nicely demonstrates the importance of the recursive technique described in the notes: Carefully write the specs for each program, believe by magic that the programs work, write the programs calling themselves as if they already work, and make sure that as you recurse the instance being inputted gets "smaller".

The Grammar: As an example, we will be considering a very simple grammar that considers expressions over $\times$ and $+$.

exp $\Leftarrow$ term
    $\Leftarrow$ term + exp
term$\Leftarrow$ fact
    $\Leftarrow$ fact * term
fact $\Leftarrow$ int
    $\Leftarrow$ (exp)

A Derivation of a String:

```
s = ( ( 2 + 42 ) * ( 5 + 12 ) + 987 * 7 * 123 + 15 * 54 )
```

```
|-exp-----------------------------------------|
|-term----------------------------------------|
|-fact----------------------------------------|
( |-exp----------------------------------------| )
  |-term----------------| + |-exp----------------|
  |-fact---| * |-term---|   |-term------| + |-exp-|
  ( |-ex-| )   |-fac---|     f  * |-t---|   |-term-|
    t + e      ( |-ex-| )   978   f * t     f  * t
    f   t        t + e             7   f     15   f
    2   f        f   t                 123        54
        42       5   f
                     12
s = ( ( 2 + 42 ) * ( 5 + 12 ) + 987 * 7 * 123 + 15 * 54 )
```

A Parsing of an Expression: The following are different forms that a parsing of an expression could take.

- A binary tree data structure with each internal node representing either '*' or '+' and leaves representing integers.
- A text based picture of the above described tree.

```
s = ( ( 2 + 42 ) * ( 5 + 12 ) + 987 * 7 * 123 + 15 * 54 ) =
p =
                        |-- 2
                   |- + -
                   |    |-- 42
              |- * -
              |    |    |-- 5
              |    |- + -
              |         |-- 12
         -- + -
              |         |-- 987
              |    |- * -
              |    |    |    |-- 7
              |    |    |- * -
              |    |         |-- 123
              |- + -
                   |    |-- 15
                   |- * -
                        |-- 54
```

- A string with more brackets to indicated the internal structure.

```
s = ( (2+42) * (5+12)   +   987*7*123    +  15*54  )
p = (((2+42) * (5+12))  + ((987*(7*123)) + (15*54)))
```

- An integer evaluation of the expression.

```
s = ( ( 2 + 42 ) * ( 5 - 12 ) + 987 * 7 * 123 + 15 * 54 )
p = 851365
```

Another Example: Consider the following grammar. The start symbol is $\langle Op \rangle$.

$$
\begin{array}{rcl}
\langle Op \rangle & \rightarrow & \langle If \rangle \mid \langle IfElse \rangle \mid \langle While \rangle \mid \langle Assign \rangle \mid \{\langle Ops \rangle\} \\
\langle Ops \rangle & \rightarrow & \langle Op \rangle \mid \langle Op \rangle \, \langle Ops \rangle \\
\langle If \rangle & \rightarrow & if(\langle Boolean \rangle) \; then \; \langle Op \rangle \\
\langle IfElse \rangle & \rightarrow & if(\langle Boolean \rangle) \; then \; \langle Op \rangle \; else \; \langle Op \rangle \\
\langle While \rangle & \rightarrow & while(\langle Boolean \rangle) \; \langle Op \rangle
\end{array}
$$

$$
\begin{array}{rcl}
\langle Assign \rangle & \rightarrow & \langle Var \rangle = \langle Equ \rangle \\
\langle Boolean \rangle & \rightarrow & \langle Equ \rangle = \langle Equ \rangle \mid \langle Equ \rangle < \langle Equ \rangle \\
\langle Equ \rangle & \rightarrow & \langle Term \rangle \mid \langle Term \rangle + \langle Equ \rangle \\
\langle Term \rangle & \rightarrow & \langle Factor \rangle \mid \langle Factor \rangle \times \langle Term \rangle \\
\langle Factor \rangle & \rightarrow & \langle Var \rangle \mid \langle Num \rangle \mid (\langle Equ \rangle) \\
\langle Var \rangle & \rightarrow & i \mid j \mid k \mid x \mid y \mid z \\
\langle Num \rangle & \rightarrow & \langle digit \rangle \mid \langle digit \rangle \langle Num \rangle \\
\langle digit \rangle & \rightarrow & 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9
\end{array}
$$

Below are three code fragments. View each as one entire string. Ignore spaces and line breaks. For each, do the following. If possible give a parsing for it. If not explain why. If possible give a second parsing for it. Is there anything about this code fragment that indicates a potential danger for programmers? How do the C and the JAVA compilers deal with these problems? How do programmers deal with these problems?

(a)
```
if(y = 3 * x + 7) then {
        i = 0
        while(i < 10)
                y = (y + 10) + 5
                i = i + 1
}
```

(b)
```
if(i = y) then
        if(i = x) then
                i = 0
        else
                y = 7
```

(c)
```
y = (3 * x + 7) * 53
z = ((2 * 3 * 4 + 5 + 6) + 7)
```

- Answer: The parsing is straight forward except that the line $i = i+1$ will not be within the *while* loop. One operation after a *while* or an *if* does not need brackets, but two does. Programmers should get into the habit of always putting brackets { } around even a single *op*. Someone may latter add a second *op* and not notice that the brackets are not there.

- Answer: There are two parsings. One associates the *else* with the first *if* and the other with the second *if*. The compiler defaults to the second. Again being in the habit of always putting brackets { } around even a single command after an *if* would solve this problem.

- Answer: This has no parsing. It consists of two *op*s without a surrounding bracket.

5.1 Proving Which Language is Generated

Consider the grammar G

$$S \rightarrow aSb \mid bSa \mid SS \mid \epsilon$$

The goal of this question is to formally prove that this grammar generates the language $L = \{s \mid \#$ of a's in $s = \#$ of b's in $s\}$. This involves two steps.

$\Rightarrow$ Prove "G only generates strings that are in L." See (a).

$\Leftarrow$ Prove "G generates all the strings in L," i.e., "If $s \in L$, then G generates it." See (d).

1. Formally prove "if the string s can be generated by G, then it is in L". The proof will be by strong induction on the length $|s|$ of the string. As with all proofs by strong induction, the format is as follows:

   - (1 apple) For each integer $n \geq 0$, define $H(n)$ to be the boolean statement "_____."
     - Answer: "If the string s has length n and can be generated by G, then it is in L."
   - (1 apple) $H(0)$ is true because _____.

- Answer: the only string of length zero is $s = \epsilon$, which can be generated by G and is in L.

- (4 apples) Let n be an arbitrary integer bigger than zero. By way of strong induction, assume that each of $H(0), H(1), H(2), \ldots, H(n-1)$ is true. Using these assumptions, we prove $H(n)$ as follows. _____.
  - Answer: Let s be an arbitrary string of length n that can be generated by G. Let P be one of the shortest parsings of s. The first step of P must be one of the following.

    $S \to aSb$**:** In this case, s must be $s = as_1b$ where s_1 has length $n-2$ and can be generated by G. Hence, by $H(n-2)$, s_1 is in L. Because s_1 has the same number of a's and b's, it follows that $s = as_1b$ also has the same number of a's and b's and hence $s \in L$.

    $S \to bSa$**:** Same as above.

    $S \to SS$**:** In this case, s must be $s = s_1s_2$ where s_1 has length i, s_2 has length $n-i$, and both s_1 and s_2 can be generated by G. Hence, by $H(i)$ and $H(n-i)$, both s_1 and s_2 are in L. Because both s_1 and s_2 have the same number of a's and b's, it follows that $s = s_1s_2$ also has the same number of a's and b's and hence $s \in L$.
    (Glitch: If $i = n$, then $H(i) = H(n)$ has not been assumed. Similarly, if $i = 0$, then $H(n-i) = H(n)$ has not been. If $i = n$, then the parsing P begins as follows. $S \Rightarrow SS \Rightarrow S\epsilon$. Removing these steps gives a shorter parsing that generates the same string s. This contradict the fact that P is the shortest parsing. Hence, $i \neq 0$. Similarly $i \neq n$.)

- (1 apple) By way of strong induction we can conclude that $\forall n \geq 0\ H(n)$. From this the statement, "if the string s can be generated by G, then it is in L" follows because _____.
  - Answer: if it is true for any s of each length, then it is true for any s.

2. Let $s \in \{a, b\}^*$ be an arbitrary string. Let $n = |s|$. Define F_s to be the following function. For each integer $i \in [0, n]$, define $F_s(i)$ to be (# of a's in the first i characters of s) − (# of b's in the first i characters of s). For example, $F_{abba}(3) = -1$. To make the function $F_s(x)$ continuous for all real values x interpolate between $F_s(i)$ and $F_s(i+1)$ with a diagonal line.

   Note that for every string s, $F_s(0) = 0$. Call this graph property (i).

   (a) (2 apples) Graph F_{abbbaa} and F_{aabab}.

   (b) (1 apple) If $s \in L$, then what property does F_s have? Call this graph property (ii).
   - Answer: Then F_s ends at zero, ie $F_s(n) = 0$.

   (c) (2 apples) If $s \in L$ and s begins and ends with the same character, then what property does F_s have? Call this graph property (iii).
   - Answer: Define property (iii) to be property that $F_s(1)$ and $F_s(n-1)$ have opposite signs, i.e. one positive and the other negative.
     If s begins with an a, then F_s starts at $F_s(0) = 0$ and goes up to $F_s(1) = 1$. If s ends with an a, then F_s ends by going up from $F_s(n-1) = -1$ to $F_s(n) = 0$. Note that this means that it has property (iii).
     Similarly, if s begins with a b, then F_s starts at $F_s(0) = 0$ and goes down to $F_s(1) = -1$. If s ends with a b, then F_s ends by going down from $F_s(n-1) = 1$ to $F_s(n) = 0$. Again, it has property (iii).

   (d) (2 apples) State a graph property (iv) of F_s. Prove that if F_s has properties (i), (ii), and (iv), then s can be split into two parts $s = s_1s_2$ such that both s_1 and s_2 are in L and neither s_1 nor s_2 is the empty string.
   - Answer: Define property (iv) to be property that there is an integer i such that $i \neq 0$, $i \neq n$, and $F_s(i) = 0$.
     Suppose that F_s properties (i), (ii), and (iv). Let s_1 be the first i characters of s and s_2 be the last $n-i$ characters.
     Note that because $i \neq 0$ and $i \neq 1$ it follows that neither s_1 nor s_2 are the empty string.

Because $F_s(0) = 0$ and $F_s(i) = 0$, it follows by definition that s_1 has the same number of a's and b's and hence is in L. Similarly, because $F_s(i) = 0$ and $F_s(n) = 0$, it follows by definition that s_2 has the same number of a's and b's and hence is in L.

(e) (2 apples) Prove that any function that has graph property (iii) also has graph property (iv).

- Answer: If $F_s(1)$ and $F_s(n-1)$ have opposite signs and is continuous, then by the Mean Value Theorem from Calculus there must be an x in between for which $F_s(x) = 0$. All we need now is to prove that x is an integer. Note that for every integer i, the function $F_s(x)$ either increases or decreases by one between $F_s(i)$ and $F_s(i+1)$. Therefore, the function $F_s(x)$ takes on integer values only on integers values. If follows that if $F_s(x) = 0$, then x is an integer. This x is the integer i required for property (iv).

3. (2 apples) Prove the following lemma. (Hint: Even if you were not able to answer the questions in (b), you may use the statements in the questions to prove this lemma. Do not leave any gaps in your logic.)
Lemma: If $s \in L$ and begins and ends with the same character, then s can be split into two parts $s = s_1 s_2$ such that both s_1 and s_2 are in L and neither s_1 nor s_2 is the empty string.

- Answer: Proof:
  - All functions F_s have property (i).
  - Because $s \in L$, by question (bii), F_s has property (ii).
  - Because $s \in L$ and s begins and ends with the same character, by question (biii), F_s has property (iii).
  - Because F_s has property (iii), by question (bv), it has property (iv).
  - Because F_s has properties (i), (ii), and (iv), by question (biv), s can be split into two parts $s = s_1 s_2$ such that both s_1 and s_2 are in L and neither s_1 nor s_2 is the empty string.

4. (4 apples) Construct a recursive algorithm whose input is a string $s \in L$ and whose output is a parsing of s by the grammar. Note if $s \notin L$, then what the algorithm does is unspecified. Given an instance, $s \in L$, the algorithm will have a number of cases. All but one of these cases proceeds as follows.

- Assume that your friends can solve any instance to the same problem as long as it is strictly smaller then your instance.
- From your s construct some subinstances, ie construct one or more strings s_1 and s_2. Be sure to prove that these strings are in L and are shorter than s.
- Have your friends give you a parsing for each of your strings s_1 and s_2.
- Construct a parsing for s from your friends' parsings.

(Hint: Even if you were not able to answer question (c), you may use its statement to construct this algorithm.)

- Answer: Given an instance, $s \in L$, the recursive algorithm is as follows. There are four cases:
  $s = \epsilon$: The parsing is $S \Rightarrow \epsilon$.
  s begins with an a and ends with a b: Split s into $s = as_1b$. Because $s = as_1b$ has the same number of a's and b's, it follows that s_1 also has the same number of a's and b's, and hence $s_1 \in L$. Ask your friend to give you a parsing for s_1. Your parsing will be $S \Rightarrow aSb \Rightarrow^* as_1b = s$, where the $\Rightarrow^*$ follows your friends parsing.
  s begins with a b and ends with an a: Same as above.
  s begins and ends with the same character: By the lemma we know that s can be split into two parts $s = s_1 s_2$ such that both s_1 and s_2 are in L. Because neither are empty, both are strictly smaller than s. Ask one friend to give you a parsing for s_1 and another friend to give you a parsing of p_2. Your parsing will be $S \Rightarrow SS \Rightarrow^* s_1 S \Rightarrow^* s_1 s_2 = s$, where the first $\Rightarrow^*$ follows your first friends parsing and the second follows that of your second friend.

5. (2 apples) Trace your algorithm to produce a parsing for $s = babbaa$.

- Answer:

  Note $babbaa$ begins with a b and ends with an a. Hence its parsing is $S \Rightarrow bSa$, where the S must generate $abba$.
  Note $abba$ begins with and ends with the same character. We split it into ab and ba both in L. Hence its parsing is $S \Rightarrow SS$, where the first S must generate ab and the second ba.
  Note ab begins with an a and ends with a b. Hence its parsing is $S \Rightarrow aSb$, where the S must generate ϵ.
  ϵ is generated with $S \Rightarrow \epsilon$.
  Note ba begins with a b and ends with an a. Hence its parsing is $S \Rightarrow bSa$, where the S must generate ϵ.
  ϵ is generated with $S \Rightarrow \epsilon$.

  In conclusion the parsing is $S \Rightarrow bSa \Rightarrow bSSa \Rightarrow baSbSa \Rightarrow ba\epsilon bSa = babSa \Rightarrow babbSaa \Rightarrow babb\epsilon aa = babbaa$

5.2 Pumping Lemma

Use the pumping lemma to show that $L = \{a^n b^m c^n d^m \mid m, n \geq 0\}$ is not context free.

- Consider the following strategy for the game G_L.
  - The adversary gives us the pumping length p.
  - We choose the string $s =$ _____.
    - Answer: $s = a^p b^p c^p d^p$
  - The adversary chooses a split $s = uvxyz$ such that $|vy| > 0$ and $|vxy| \leq p$.
  - We separately consider the following cases _____.
    - Answer: Because $|vxy| \leq p$, the string vxy can only be contained in one of the four blocks of s or within two consecutive blocks.
  - We set $i =$ _____.
    - Answer: We set $i = 0$. Because $|vy| > 0$, we know that s has some of its characters removed to form uxz. More over, we know that the characters removed are in at most two consecutive blocks.
  - We prove $uv^i xy^i z \notin L$ as follows _____.
    - Answer: Consider one of the four blocks of s that has some of its characters removed. Call it block B. There is another block of s that is supposed to be the same size as this one. Call it block B'. Note B and B' are not consecutive to each other. Hence, they could not both have changed. Hence, after the change they cannot still be the same size. Hence, $uxz \notin L$.
  - In each case we win the game.
- Hence, there is no strategy with which the adversary can win.
- Hence, L is not context free.

5.3 Parsing with Context-Free Grammars

An important computer science problem is parsing a string according a given context-free grammar. A *context-free grammar* is a means of describing which strings of characters are contained within a particular language. It consists of a set of rules and a start *non-terminal* symbol. Each rule specifies one way of replacing a non-terminal symbol in the current string with a string of terminal and non-terminal symbols. When the resulting string consists only of terminal symbols, we stop. We say that any such resulting string has been *generated* by the grammar.

Context-free grammars are used to understand both the syntax and the semantics of many very useful languages, such as mathematical expressions, JAVA, and English. The *syntax* of a language indicates which strings of tokens are valid sentences in that language. The *semantics* of a language involves the meaning associated with strings. In order for a compiler or natural language "recognizers" to determine what a string means, it must *parse* the string. This involves deriving the string from the grammar and, in doing so, determining which parts of the string are "noun phrases", "verb phrases", "expressions", and "terms."

Some context-free grammars have a property called *look ahead one*. Strings from such grammars can be parsed in linear time by what I consider to be one of the most amazing and magical recursive algorithms. This algorithm is presented in this chapter. It demonstrates very clearly the importance of working within the friends level of abstraction instead of tracing out the stack frames: Carefully write the specifications for each program, believe by magic that the programs work, write the programs calling themselves as if they already work, and make sure that as you recurse the instance being input gets "smaller."

In Section **??** we will analyze an elegant dynamic-programming algorithm given that parses a string from any context-free grammar, not just look ahead one, in $\Theta(n^3)$ time.

The Grammar: We will look at a very simple grammar that considers expressions over $\times$ and $+$. In this grammar, a *factor* is either a simple integer or a more complex expression within brackets; a term is one or more factors multiplied together; and an expression is one or more terms added together. More precisely:

exp $\Rightarrow$ term
$\Rightarrow$ term + term + ... + term

term$\Rightarrow$ fact
$\Rightarrow$ fact * fact * ... * fact

fact $\Rightarrow$ int
$\Rightarrow$ (exp)

Non-Terminals, Terminals, and Rules: More generally, a grammar is defined by a set of *non-terminals*, a set of *terminals*, a *start non-terminal*, and a set of rules. Here the non-terminals are "exp", "term", and "fact." The terminals are integers, the character "+", and the character "*." The start non-terminal is "exp." Above is the list of rules for this grammar.

A Derivation of a String: A grammar defines the *language* of strings that can be derived in the following way. A derivation of a string starts with the start the non-terminal. Then each rule, like those above, say that you can replace the non-terminal on the left with the string of terminals and non-terminals on the right. The following is an example.

```
|-exp-------------------------------------------------------|
|-t-| + |-term----------------------------------------------|
f * f   |-fact----------------------------------------------|
6   8   ( |-exp-----------------------------------------| )
          |-term----------------| + |-term------| + |-term-|
```

```
                |-fact---| * |-fact---|     f * f *   f    f *  f
                ( |-ex-| )   ( |-ex-| )   987   7   123   15   54 )
                  t + t       t + t
                  f   f       f + f
                  2   42      5   12
s = 6 * 8 + ( ( 2 + 42 ) * ( 5 + 12 ) + 987 * 7 * 123 + 15 * 54 )
```

A Parsing of an Expression: Let s be a string consisting of terminals. A parsing of this string is a tree. Each internal node of the tree is labeled with a non-terminal symbol, the root with the start non-terminal. Each internal node must correspond to a rule of the grammar. For example, for rule $A \Rightarrow BC$, the node is labeled A and its two children are labeled B and C. The leafs of the tree read left to right give the input string s of terminals. The following is an example.

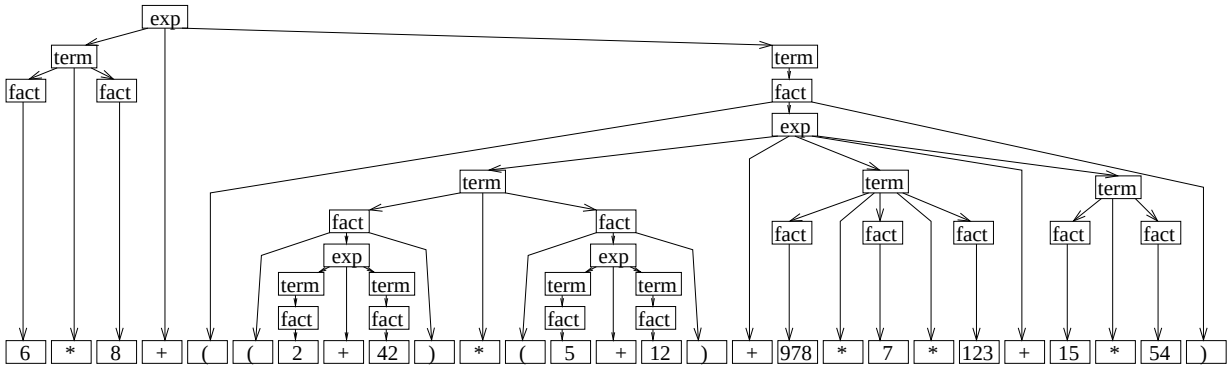


Figure 5.1: A parse tree for the string $s = 6 * 8 + ((2 + 42) * (5 + 12) + 987 * 7 * 123 + 15 * 54)$.

The Parsing Abstract Data Type: The following is an example where it is useful not to give the full implementation details of an abstract data type. If fact, we will even leave the specification of parsing structure open for the implementer to decide.

For our purposes, we will only say the following: When p is a variable of type parsing, we will use "p=5" to indicate that it is assigned a parsing of the expression "5". We will go on to *overload* the operations $*$ and $+$ as operations that join two parsings into one. For example, if p_1 is a parsing of the expression "2*3" and p_2 of "5*7", then we will use $p = p_1 + p_2$ to denote a parsing of expression "2*3 + 5*7".

The implementer defines the structure of a parsing by specifying in more detail what these operations do. For example, if the implementer wants a parsing to be a binary tree representing the expression, then $p_1 + p_2$ would be the operation of constructing a binary tree with the root being a new '+' node, the left subtree being the binary tree p_1, and the right subtree being the binary tree p_2. On the other hand, if the implementer wants a parsing to be simply an integer evaluation of the expression, then $p_1 + p_2$ would be the integer sum of the integers p_1 and p_2.

Specifications for the Parsing Algorithm:

Precondition: The input consists of a string of tokens s. The possible tokens are the characters '*' and '+' and arbitrary integers. The tokens are indexed as $s[1], s[2], s[3], \ldots, s[n]$.

Postcondition: If the input is a valid "expression" generated by the grammar, then the output is a "parsing" of the expression. Otherwise, an error message is output.

The algorithm consists of one routine for each *non-terminal* of the grammar: *GetExp*, *GetTerm*, and *GetFact*.

Specifications for *GetExp*:

Precondition: The input of *GetExp* consists of a string of tokens s and an index i that indicates a starting point within s.

Postcondition: The output consists of a parsing of the longest substring $s[i], s[i+1], \ldots, s[j-1]$ of s that starts at index i and is a valid expression. The output also includes the index j of the token that comes immediately after the parsed expression.

If there is no valid expression starting at $s[i]$, then an error message is output.

The specifications for *GetTerm* and *GetFact* are the same as for *GetExp*, except that they return the parsing of the longest term or factor starting at $s[i]$ and ending at $s[j-1]$.

Examples of *GetExp*, *GetTerm*, and *GetFact*:

```
GetExp:
   s = ( ( 2 * 8 + 42 * 7 ) * 5 + 8 )
       i                              j  p = ( ( 2 * 8 + 42 * 7 ) * 5 + 8 )
         i                          j     p =   ( 2 * 8 + 42 * 7 ) * 5 + 8
           i             j               p =     2 * 8 + 42 * 7
                   i     j               p =             42 * 7
                               i     j    p =                         5 + 8

   GetTerm:
   s = ( ( 2 * 8 + 42 * 7 ) * 5 + 8 )
       i                              j  p = ( ( 2 * 8 + 42 * 7 ) * 5 + 8 )
         i                       j        p =   ( 2 * 8 + 42 * 7 ) * 5
           i     j                        p =     2 * 8
                   i     j                p =             42 * 7
                               i j        p =                         5

   GetFact:
   s = ( ( 2 * 8 + 42 * 7 ) * 5 + 8 )
       i                              j  p = ( ( 2 * 8 + 42 * 7 ) * 5 + 8 )
         i                  j            p =   ( 2 * 8 + 42 * 7 )
           i j                            p =     2
                   i  j                   p =             42
                               i j        p =                         5
```

Reasoning for *GetExp*: Consider some input string s and some index i. The longest substring $s[i], \ldots, s[j-1]$ that is a valid expression consists of some number of terms added together. In all of these case, it begins with a term. By magic, assume that the *GetTerm* routine already works. Calling $GetTerm(s, i)$ will return p_{term} and j_{term}, where p_{term} is the parsing of this first term and j_{term} indexes the token immediately after this term. Specifically, if the expression has another term then j_{term} indexes the '+' that is between these terms. Hence, we can determine whether there is another term by checking $s[j_{term}]$. If $s[j_{term}] =$ '+', then *GetExp* will call *GetTerm* again to get the next term. If $s[j_{term}]$ is not a '+' but some other character, then *GetExp* is finished reading in all the terms. *GetExp* then constructs the parsing consisting of all of these terms added together.

The reasoning for *GetTerm* is the same.

***GetExp* Code:**

algorithm $GetExp(s, i)$

⟨*pre−cond*⟩: s is a string of tokens and i is an index that indicates a starting point within s.

⟨*post−cond*⟩: The output consists of a parsing p of the longest substring $s[i], s[i+1], \ldots, s[j-1]$ of s that starts at index i and is a valid expression. The output also includes the index j of the token that comes immediately after the parsed expression.

begin
if ($i > |s|$) return "Error: Expected characters past end of string." end if
$\langle p_{\langle term,1\rangle}, j_{\langle term,1\rangle}\rangle = GetTerm(s, i)$
$k = 1$
loop
⟨loop−invariant⟩: The first k terms of the expression have been read.
exit when $s[j_{\langle term,k\rangle}] \neq$ '+'
$\langle p_{\langle term,k+1\rangle}, j_{\langle term,k+1\rangle}\rangle = GetTerm(s, j_{\langle term,k\rangle} + 1)$
$k = k + 1$
end loop
$p_{exp} = p_{\langle term,1\rangle} + p_{\langle term,2\rangle} + \ldots + p_{\langle term,k\rangle}$
$j_{exp} = j_{\langle term,k\rangle}$
return $\langle p_{exp}, j_{exp}\rangle$
end algorithm

***GetTerm* Code:**

algorithm $GetTerm\,(s, i)$

⟨pre−cond⟩: s is a string of tokens and i is an index that indicates a starting point within s.

⟨post−cond⟩: The output consists of a parsing p of the longest substring $s[i], s[i+1], \ldots, s[j-1]$ of s that starts at index i and is a valid term. The output also includes the index j of the token that comes immediately after the parsed term.

begin
if ($i > |s|$) return "Error: Expected characters past end of string." end if
$\langle p_{\langle fact,1\rangle}, j_{\langle fact,1\rangle}\rangle = GetFact(s, i)$
$k = 1$
loop
⟨loop−invariant⟩: The first k facts of the term have been read.
exit when $s[j_{\langle fact,k\rangle}] \neq$ '*'
$\langle p_{\langle fact,k+1\rangle}, j_{\langle fact,k+1\rangle}\rangle = GetFact(s, j_{\langle fact,k\rangle} + 1)$
$k = k + 1$
end loop
$p_{term} = p_{\langle fact,1\rangle} * p_{\langle fact,2\rangle} * \cdots * p_{\langle fact,k\rangle}$
$j_{term} = j_{\langle fact,k\rangle}$
return $\langle p_{term}, j_{term}\rangle$
end algorithm

Reasoning for *GetFact*: The longest substring $s[i], \ldots, s[j-1]$ that is a valid factor has one of the following two forms:

fact ⇒ int
fact ⇒ (exp)

Hence, we can determine which form the factor has by testing $s[i]$.

If $s[i]$ is an integer, then we are finished. p_{fact} is a parsing of this single integer $s[i]$ and $j_{fact} = i + 1$. The +1 moves the index past the integer.

If $s[i] =$ '(', then for s to be a valid factor there must be a valid expression starting at $j_{term}+1$, followed by a closing bracket ')'. We can parse this expression with $GetExp(s, j_{term} + 1)$, which returns p_{exp} and j_{exp}. The closing bracket after the expression must be in $s[j_{exp}]$. Our parsed factor will be $p_{fact} = (p_{exp})$ and $j_{fact} = j_{exp} + 1$. The +1 moves the index past the ')'.

If $s[i]$ is neither an integer nor a '(', then it cannot be a valid factor. Give a meaningful error message.

***GetFact* Code:**

algorithm $GetFac(s, i)$

⟨*pre−cond*⟩: s is a string of tokens and i is an index that indicates a starting point within s.

⟨*post−cond*⟩: The output consists of a parsing p of the longest substring $s[i], s[i+1], \ldots, s[j-1]$ of s that starts at index i and is a valid factor. The output also includes the index j of the token that comes immediately after the parsed factor.

```
begin
    if (i > |s|) return "Error: Expected characters past end of string." end if
    if (s[i] is an int)
        p_fact = s[i]
        j_fact = i + 1
        return ⟨p_fact, j_fact⟩
    else if (s[i] = '(')
        ⟨p_exp, j_exp⟩ = GetExp(s, i + 1)
        if (s[j_exp] = ')')
            p_fact = (p_exp)
            j_fact = j_exp + 1
            return ⟨p_fact, j_fact⟩
        else
            Output "Error: Expected ')' at index j_exp"
        end if

    else
        Output "Error: Expected integer or '(' at index i"
    end if
end algorithm
```

Tree of Stack Frames: $GetExp$ calls $GetTerm$, which calls $GetFact$, which may call $GetExp$, and so on. If one were to draw out the entire tree of stack frames showing who calls who, this would exactly mirror the parse tree that created.

Running Time: We prove that the running time of this entire computation is linear in the size to the parse tree produced and that this is linear in the size $\Theta(n)$ of the input string.

To prove the first, it is sufficient to prove that the running time of each stack frame is either constant or is linear in the number of children of the node in the parse tree that this stack frame produces. For example, if stack frame for $GetFact$ finds an integer than its node in the parse tree has no children, but $GetFact$ uses only a constant amount of time. In contrast, if a stack frame for $GetExp$ reads in t terms, then its running time will be some constant times t and its node in the parse tree will have t children.

We now prove that the size to the parse tree produced is linear in the size $\Theta(n)$ of the input string. If the grammar is such that every non-terminal goes to at least one terminal or at least two non-terminals, then each node in the parse tree is either a leaf or has at least two children. If follows that the number of nodes in the parse tree will be at most some constant times the number of leaves, which is the size of the input string. In our grammar, however, an expression might go to a single term, which can go to a single facor. This creates a little path of out degree one. It cannot, however, be longer than this because a factor is either a leaf or has three childeren, one is "(", the second an expression, and the third ")". Such little paths can only incress the size of the parse tree by a factor of three.

In conclusion, the running time is $\Theta(n)$.

Proof of Correctness: To prove that a recursive program works, we must consider the "size" of an instance. The routine needs only consider the postfix $s[i], s[i+1], \ldots$, which contains $(|s|-i+1)$ characters. Hence, we will define the size of instance $\langle s, i\rangle$ to be $|\langle s, i\rangle| = |s| - i + 1$.

Let $H(n)$ be the statement "Each of $GetFac$, $GetTerm$, and $GetExp$ work on instances $\langle s, i\rangle$ when $|\langle s, i\rangle| = |s| - i + 1 \leq n$." We prove by way of induction that $\forall n \geq 0,\ H(n)$.

If $|\langle s, i\rangle| = 0$, then $i > |s|$: There is not a valid expression/term/factor starting at $s[i]$, and all three routines return an error message. It follows that $H(0)$ is true.

If $|\langle s, i\rangle| = 1$, then there is one remaining token: For this to be a factor, term, or expression, this token must be a single integer. $GetFac$ is written to give the correct answer in this situation. $GetTerm$ gives the correct answer, because it calls $GetFac$. $GetExp$ gives the correct answer, because it calls $GetTerm$ which in turn calls $GetFac$. It follows that $H(1)$ is true.

Assume $H(n-1)$ is true, that is, that "Each of $GetFac$, $GetTerm$, and $GetExp$ work on instances of size at most $n-1$."

Consider $GetFac(s, i)$ on an instance of size $|s| - i + 1 = n$. It makes at most one subroutine call, $GetExp(s, i+1)$. The size of this instance is $|s| - (i+1) + 1 = n - 1$. Hence, by assumption this subroutine call returns the correct answer. Because all of $GetFac(s, i)$'s subroutine calls return the correct answer, it follows that $GetFac(s, i)$ works on all instances of size n.

Now consider $GetTerm(s, i)$ on an instance of size $|s| - i + 1 = n$. It calls $GetFac$ some number of times. The input instance for the first call $GetFac(s, i)$ still has size n. Hence, the induction hypothesis $H(n-1)$ does NOT claim that it works. However, the previous paragraph proves that this routine does in fact work on instances of size n. The remaining calls are on smaller instances.

Finally, consider $GetExp(s, i)$ on an instance $\langle s, i\rangle$ of size $|s| - i + 1 = n$. We use the previous paragraph to prove that is first subroutine call $GetTerm(s, i)$ works.

In conclusion, all three work on all instances of size n and hence on $H(n)$. This completes the induction step.

Look Ahead One: A grammar is said to be *look ahead one* if, given any two rules for the same non-terminal, the first place that the rules differ is a difference in a terminal. This feature allows our parsing algorithm to look only at the next token in order to decide what to do next. Thus the algorithm runs in linear time.

An example of a good set of rules would be:

A ⇒ B 'b' C 'd' E
A ⇒ B 'b' C 'e' F
A ⇒ B 'c' G H

An example of a bad set of rules would be:

A ⇒ B C
A ⇒ D E

With such a grammar, you would not know whether to start parsing the string as a B or a D. If you made the wrong choice, you would have to back up and repeat the process.

Exercise: Consider $s =$ "(((1) * 2 + 3) * 5 * 6 + 7)".

1. Give a derivation of the expression s.
2. Draw the tree structure of the expression s.
3. Trace out the execution of your program on $GetExp(s, 1)$. In other words, draw a tree with a box for each time a routine is called. For each box, include only whether it is an expression, term, or factor and the string $s[i], \ldots, s[j-1]$ that is parsed.

Chapter 6

Steps and Possible Bugs in Proving that CLIQUE is NP-Complete

A language is said to be *NP-complete* if (1) it is in NP and (2) every language in NP can be polynomially reduced to it. To prove (2), it is sufficient to prove that that it is at least as hard as some problem already known to be NP-complete. For example, once one knows that 3-SAT is NP-complete, one could prove that CLIQUE is NP-complete by proving that it is in NP and that 3-SAT $\leq_p$ CLIQUE.

One helpful thing to remember is the **type** of everything. For example, ϕ is a 3-formula, ω is an assignment to a 3-formula, G_ϕ is a graph, k is an integer, and S is a subset of the nodes of the graph. Sipser's book treats each of these as binary strings. This makes it easier to be more formal but harder to be more intuitive. We sometimes say that ϕ is an *instance* of the 3-SAT problem, the problem being to decide if $\phi \in$ 3-SAT. (Technically we should be writing $< \phi >$ instead of ϕ in the above sentence, but we usually omit this.)

It is also helpful to remember what we know about these objects. For example, ϕ may or may not be satisfiable, ω may or may not satisfy ϕ, G_ϕ may or may not have a clique of size k, and S may or may not be a clique of G_ϕ of size k.

To prove that 3-SAT is in NP, Jeff personally would forget about Non-deterministic Turing machines because they confuse the issue. Instead say, "A witness that a 3-formula is satisfiable is a satisfying assignment. If an adversary gives me a 3-formula ϕ and my God figure gives me an assignment ω to ϕ, I can check in poly-time whether or not ω in fact satisfies ϕ. So 3-SAT is in NP."

The proof that $L_1 \leq L_2$ has a standard format. Within this format there are five different places that you must plug in something. Below are the five things that must be plugged in. To be more concrete we will use specific languages 3-SAT and CLIQUE.

To prove that 3-SAT $\leq_p$ CLIQUE, the necessary steps are the following:

1. Define a function f mapping possible inputs to 3-SAT to possible inputs to CLIQUE. In particular, it must map each 3-formula ϕ to a $\langle G_\phi, k\rangle$ where G_ϕ is a graph and k is an integer. Moreover, you must prove that f is poly-time computable.

   In the following steps (2-5) we must prove that f is a valid mapping reduction, i.e. that $\phi \in$ 3-SAT if and only if $\langle G_\phi, k\rangle \in$ CLIQUE.

2. Define a function g mapping possible witnesses for 3-SAT to possible witnesses to CLIQUE. In particular, it must map each assignment ω of ϕ to subsets S of the nodes of G_ϕ, where $\langle G_\phi, k\rangle = f(\phi)$. Moreover, you must prove that the mapping is well defined.

3. Prove that if ω satisfies ϕ, then $S = g(\omega)$ is a clique of G_ϕ of size at least k.

4. Define a function h mapping possible witnesses for CLIQUE to possible witnesses to 3-SAT. In particular, it must map each k-subset S of the nodes of G_ϕ (where $\langle G_\phi, k\rangle = f(\phi)$) to assignments ω of ϕ and prove that the mapping is well defined. (Though they often are, g and h do not need to poly-time computable. In fact, they do not need to be computable at all.)

5. Prove that if S is a clique of G_ϕ of size k, then $\omega = h(c)$ satisfies ϕ.

When reducing from 3-SAT, the instance $f(\phi)$ typically has a gadget for each variable and a gadget for each clause. The variable gadgets are used in step 4 to assign a definite value to each variable x. The clause gadgets are used in step 5 to prove that each clause is satisfied.

Proof of 3-SAT $\leq_p$ CLIQUE:

To prove that f is a valid mapping reduction we must prove that for any input ϕ
(A) if $\phi \in$ 3-SAT then $\langle G_\phi, k\rangle \in$ CLIQUE
and (B) if $\phi \notin$ 3-SAT then $\langle G_\phi, k\rangle \notin$ CLIQUE.

Once we have done the above five steps we are done because:

(A) Suppose that $\phi \in$ 3-SAT, i.e. ϕ is a satisfiable 3-formula. Let ω be some satisfying assignment for ϕ. Let $S = g(\omega)$ be the subset of the nodes of G_ϕ given to us by step 2. Step 3 proves that because ω satisfies ϕ, it follows that S is a clique in G_ϕ of size k. This verifies that G_ϕ has a clique of size k and hence $\langle G_\phi, k\rangle \in$ CLIQUE.

(B) Instead of proving what was stated, we prove the contrapositive, namely that if $\langle G_\phi, k\rangle \in$ CLIQUE, then $\phi \in$ 3-SAT. Given G_ϕ has a clique of size k, let S be one such clique. Let $\omega = h(S)$ be the assignment to ϕ given to us in step 4. Step 5 proves that because S is a clique of G_ϕ of size k, $\omega = h(S)$ satisfies ϕ. Hence, ϕ is satisfiable and so $\phi \in$ 3-SAT.

This concludes the proof that 3-SAT $\leq_p$ CLIQUE.

Common Mistakes (Don't do these things)

1. **Problem:** Start defining f with "Consider a $\phi \in$ 3-SAT."
   Why: The statement "$\phi \in$ 3-SAT" means that ϕ is satisfiable. f must be defined for every 3-formula, whether satisfiable or not.
   Fix: "Consider an instance of 3-SAT, ϕ" or "Consider a 3-formula, ϕ".

2. Big **Problem:** Using the witness for ϕ (a satisfying assignment ω) in your construction of $f(\phi)$.
   Why: We don't have a witness and finding one may take exponential time.

3. **Problem:** Not doing these steps separately and clearly, but mixing them all together.
   Why: It makes it harder to follow and increases likelihood of the other problems below.

4. **Problem:** The witness-to-witness function is not well-defined or is not proved to be well-defined.
   Example: Define an assignment ω from a set of nodes S as follows. Consider a clause gadget. If the node labeled x is in S than set x to be true.
   Problem: The variable x likely appears in many clauses and some of the corresponding nodes may be in S and some may not.
   Fix: Consider the variable gadget when defining the value of x.

5. **Problem:** When you define the assignment $\omega = h(S)$, you mix it within the same paragraph that you prove that it satisfies ϕ.
   Danger: You may say, "Consider the clause $(x \wedge y \wedge z)$. Define x to be true. Hence the clause is satisfied. Now consider the clause $(\overline{x} \wedge z \wedge q)$. Define x to be false. Hence the clause is satisfied."

6. **Problem:** Defining $\omega = h(S)$ to simply be the inverse function of $S = g(\omega)$. **Danger:** There may be some sets of nodes S that don't have the form you want and then $\omega = h(S)$ is not defined. **Why:** God gives you a set of nodes S. You are only guaranteed that it is clique of G of size k. It may not be of

the form that you want. If you believe that in order to be a clique of size k it has to be of the form you want, then you must prove this.

Chapter 7

Appendix

7.1 Existential and Universal Quantifiers. Proofs.

People are challenged proving things. It certainly helps to have good intuition about what needs to be proved and the statements that you know are true. It also helps to be able to mechanically and meticulously to be able to follow the formal proof steps. Do not skip any.

A proof is a sequence of statement in which each is either an axiom known to be true or follows logically from previous statements. It helps to clearly state your goals, write each statement on a separate line, and indent for each line how you know it is true.

To be able to understand and prove formal statements, it is important to have intuition, but it is also important to write down what is true in a formal statement. First order logic is very important and is one of the things that students consistently get wrong.

- $\forall x\ \exists y\ x + y = 5$ is true. Let x be an arbitrary real value and let $y = 5 - x$. Then $x + y = 5$.
- $\exists y\ \forall x\ x + y = 5$ is false. Let y be an arbitrary real value and let $x = 6 - y$. Then $x + y \neq 5$.

These are the steps to follow when you prove that such a statement is true or false or when you are writing such a statement yourself. Be sure to play the correct game as to who is providing what value:

Direction: Move left to right through expression.

Exists: $(\exists y)$ means that you the prover get to chose your favorite value for y. In the proof write, "Let y be the value 5.

Forall: $(\forall x)$ means an adversary chooses the worst case object for x. In the proof write, "Let x be an arbitrary value."

Truth of Predicate: After an object has been chosen for each variable, check if the predicate is true or not.

Truth of Statement: If there is a strategy for you to win, no matter how the adversary plays, then the statement is true. If there is a strategy for the adversary to win, no matter how you play, then the statement is false.

Proving False: You prove the statement false by proving that the negation of the statement is true. Note that the exists and the foralls switch and hence roles of prover and the adversary switch.

$\forall x \exists y \forall z F(x, y, z)$: The proof/game that this is true is a follows.
Let x be an arbitrary value given to me by an Adversary.
I construct the value y_x as follows .... My choice of y_x can depend on the adversary's choice of x.
Let $z_{\langle x, y_x \rangle}$ be an arbitrary value given to me by an Adversary.
I win if $F(x, y_x, z_{\langle x, y_x \rangle})$.

Examples: Use these same game ideas to help you construct statement in first order logic.

Define $P(I)$ to be the required output of the computational problem P on inputs instance I and define $A(I)$ to be the actual output of the algorithm A on inputs instance I. We want to express the statement "Problem P is computable by some algorithm."

- $\exists A$ such that A computes P.
  This is not a good first order logic statement because the word "computes" is not defined in this language.
- $\forall I,\ P(I) = A(I)$.
  This correctly expresses the fact that algorithm A gives the correct answer for problem P on every input instance I. Hence it is a correct expression for saying "A computes P." However, in the statement "Problem P is computable by some algorithm," the variable A is what is known as a *free variable* in that it is not bound to a specific object. The statement, therefore, needs either a $\forall A$ or an $\exists A$. In contrast, the variable P is bound, in that the statement says something *about* the problem P and hence P is bound to what ever object the statement is talking about. Hence, the statement needs neither a $\forall P$ nor an $\exists P$.
- $\forall I,\ \exists A,\ P(I) = A(I)$.
  This incorrectly captures the concept because it allows there to be a different algorithm for each input instance.
- $\exists A,\ \forall I,\ P(I) = A(I)$.
  This is correct. Here are a few reasons the $\exists A$ needs to come before the $\forall I$.
  - You need one algorithm that works for every instance.
  - As a game, you must first provide the algorithm before the discussion about whether it solved problem P can even begin. Once provide, you prove its correctness by testing it on all instances.
  - Think about the brackets. The statement "Problem P is computable by some algorithm" is the same as $\exists A$ such that "A computes P." The statement "A computes P" is expressed as $\forall I,\ P(I) = A(I)$. Combine these to get the correctly bracketed expression $\exists A,\ (\forall I,\ P(I) = A(I))$.

Implication: The statement $A \Rightarrow B$ is a logic statement that is either true or false.

- It means that if A is true then B is also true.
- This could be because A causes B.
- An equivalent statement, called the "counter positive," is $\neg B \Rightarrow \neg A$, because if B is not true, then A can't be true because otherwise B would be true. Hence, the statement $A \Rightarrow B$ could be true because B being false causes A to be false.
- Or maybe C causes both A and B to be true.
- Or maybe cause and effect is not involved at all.
- $A \Rightarrow B$ formally means $\neg(A$ and $\neg B)$
  "It is not true that both A and not B are true"
- Bring the negation in gives $\neg A$ or B
  If A is false, then the statement $A \Rightarrow B$ follows automatically. Similarly if B is true, then $A \Rightarrow B$ again follows.

In the proof of $A \Rightarrow B$, one officially needs to consider two cases. In the case that A is false, the statement $A \Rightarrow B$ is trivially true. In the case that A is true, one needs to do some work to prove that B is true. To be simpler, we tend to ignore the first case and simply assume A and prove B. Generally, a proof consists of a sequence of statements all that follow from the initial assumptions. When we make the new assumption that A is true, the proof is easier to read if it indents the lines for the duration of this new assumption. Then after B is proved, this indenting "stack" is popped with the conclusion that $A \Rightarrow B$. It helps to state all of your goals and conclutions in order to give the reader the heads up.

- Goal is to prove $A \Rightarrow B$.
- Assume that A is true.
  - Goal is to prove B.
  - ... proof of B.
  - Hence B is true.
- Hence $A \Rightarrow B$ is true.

Assuming Forall: If you assume that the truth of $S_1 : \forall x\ P(x)$, then at any point in time you know that $P(x')$ is true for your favorite value x'.

- Assume $S_1 : \forall x\ P(x)$
  - Construct your favorite x'.
  - Because S_1 is true for all x it must be true for your x'.
  - Hence $P(x')$ is true.

Assuming Exists: If you assume that the truth of $S_2 : \exists y\ Q(y)$, then at any point in time you can be given such a value of y.

- Assume $S_2 : \exists y\ Q(y)$.
  - Let y' be the value of y stated to exists by S_2.
  - Hence $Q(y')$ is true.

Assuming Forall Exists: If you assume that the truth of $S_3 : \forall x\ \exists y\ R(x, y)$, then for each x' you can be given a $y_{x'}$ for which $R(x', y_{x'})$ is true. In fact, S_3 being true gives you a function from values x' of x to values $y_{x'}$ of y.

- Assume $S_3 : \forall x\ \exists y\ R(x, y)$.
  - Construct your favorite x'.
  - Because S_3 is true for all x it must be true for your x'.
  - Hence $\exists y\ R(x', y)$ is true.
  - Let $y_{x'}$ be the value of y stated to exists.
  - Hence $Q(x', y_{x'})$ is true.

Example:

- Our goal is to prove $[\exists y\ \forall x\ R(x, y)] \Rightarrow [\forall x\ \exists y\ R(x, y)]$.
- Assume that $S_1 : \exists y\ \forall x\ R(x, y)$.
  - Our goal is to prove $S_2 : \forall x\ \exists y\ R(x, y)$.
  - Following the first order game, let x' be an arbitrary value.
  - It is then my job to construct a y'.
    - Let y' be the value of y stated to exists by S_1.
    - Hence $S_3 : \forall x\ R(x, y')$ is true.
  - To complete the first order game proof of S_2, I must prove that $R(x', y')$ is true.

        * Because S_3 is true for all x it must be true for our x'.
        * Hence $R(x', y')$ is true.
    - Hence by our game $S_2 : \forall x\ \exists y\ R(x, y)$ is true.
- Hence $[\exists y\ \forall x\ R(x, y)] \Rightarrow [\forall x\ \exists y\ R(x, y)]$ is also true.

Subsets: Prove $S_1 \subseteq S_2$ by proving $\forall x,\ [x \in S_1 \Rightarrow x \in S_2]$.