York University
CSE 2001 – Unit 5.0 Uncountable
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Read Jeff's notes. Read the book. Go to class. Ask lots of question. Study the slides. Work hard on solving these questions on your own. Talk to your friends about it. Talk to Jeff about it. Only after this should you read the posted solutions. Study the solutions. Understand the solutions. Memorize the solutions. The questions on the tests will be different. But the answers will be surprisingly close.

1. What is the "size" of the following sets:

i	$\{i \mid i$ is a prime integer$\}$:	Finite	Countably-Infinite	Uncountable
ii	$\{G \mid G$ is a grammar$\}$:	Finite	Countably-Infinite	Uncountable
iii	$\{L \mid L$ is a regular language$\}$:	Finite	Countably-Infinite	Uncountable
iv	$\{L \mid L$ is a language$\}$:	Finite	Countably-Infinite	Uncountable
v	all points on a 1 inch line:	Finite	Countably-Infinite	Uncountable
vi	all the atoms in the earth:	Finite	Countably-Infinite	Uncountable

2. Write pseudo code that loops over every tuple of 4 positive integers $\langle w, x, y, z\rangle$ such that each is eventually printed (and only printed once).

3. Prove that there are more real numbers than integers, i.e. $|\mathcal{R}| > |\mathcal{N}|$.

4. Rationals:

   (a) Each fraction has an infinite description, eg $\frac{1}{3} = 0.33333\ldots$. Didn't we say that this means the set of fractions $\mathcal{Q}$ is uncountable? Explain why or why not.

   (b) Study the prove that the set of real numbers is uncountable. Use the exact same proof to show that $\mathcal{Q}$ is uncountable. What if anything goes wrong in the proof?

5. Power Sets: Let U be a set of objects. In this question we will first have it be the set of positive integers and then be the set of positive reals less than one. The power set of U is the set of all subsets of U. If U is finite, then its power set has cardinality $2^{|U|}$ elements. Hence, power set of U is often denoted by $2^U = \{s \mid s \subseteq U\}$. Similarly, denote $2^U_{finite} = \{s \mid s \subseteq U$ and $|s|$ is finite$\}$. We will consider the relative sizes of U, 2^U_{finite}, and 2^U.

   (a) Recall that we say $|2^U_{finite}| \leq |U|$ if $\exists$ a function $F : 2^U_{finite} \rightarrow U$ such that $\forall s \in 2^U_{finite}, F(s) \in U$ and $\forall s, s' \in 2^U_{finite},\ s \neq s' \Rightarrow F(s) \neq F(s')$. A common way to prove $s \neq s' \Rightarrow F(s) \neq F(s')$ is to provided the inverse function F^{-1} and prove that $\forall s\ F^{-1}(F(s) = s$. (Though I did want you think about this, I don't ask you to do it.)

      i. Let $U = \mathcal{N}$ denote the set of positive integers. Then $2^{\mathcal{N}}_{finite}$ denotes the set of finite subsets of $\mathcal{N}$. Prove that $2^{\mathcal{N}}_{finite}$ is countable by defining a concrete function $F(s) = u_s$ mapping each finite subsets s of the positive integers to a unique integer u_s. Use the ascii technique given in the slides.
      - If $s = \{24, 8\}$, what is $F(s)$?
      - What happens if you don't encode the commas?
      - Does the fact that the integers in s can be put into different orders create a problem?
      - What two properties of s are key in proving that for every $s \in 2^{\mathcal{N}}_{finite}$, $F(s)$ is a finite integer?
      - Look at the ASCII table. Why might I have gotten nervous about using the decimal code instead of the hex code?

ii. Let $\mathcal{R}_{[0,1)}$ denote the set of positive reals less than one. Let $2^{\mathcal{R}_{[0,1)}}_{finite}$ denote the set of finite subsets of $\mathcal{R}_{[0,1)}$. Let $\mathcal{R}$ denote the set of reals (possibly bigger than one). Prove that $2^{\mathcal{R}_{[0,1)}}_{finite}$ has cardinality at most that of $\mathcal{R}$ by defining a concrete function $F(s) = x_s$ mapping each finite subsets s of the positive reals less than one to a unique real x_s. Be as explicit as you can. Hint: Interweave the bits. Be sure (but don't prove) that for your construction, for every $s \in 2^{\mathcal{N}}_{finite}$, $F(s)$ is valid real number and that if s and $s' \in 2^{\mathcal{R}}_{finite}$ are different then $F(s) \neq F(s')$.

(b) A Hierarchy of infinities.

i. Prove that for every set U, the cardinality of 2^U is strictly bigger than that of U, i.e. $|2^U| > |U|$. In our third definition of $|2^U| \leq |U|$, we argued that if each object $u \in U$ is able to hit at most one element $F^{-1}(u) = s \in 2^U$ and this process manages to hit every element $s \in 2^U$, then it follows that $|2^U| \leq |U|$. Conversely, we prove $|2^U| > |U|$, by proving that $\forall$ inverse functions F^{-1} from U ideally to 2^U, $\exists s_{new} \in 2^U, \forall u \in U, F^{-1}(u) \neq s_{new}$.
Your proof should use the first order logic game between the adversary and the prover. Note unlike the proof that $|R| > |N|$, the set U might not be countable and hence can't be listed and hence the diagonal can be visualized.
Hint: Woody Allen once said that he did not want to be a member of any club that would have him as a member. In this spirit, for each person, put him in the heaven club iff he is not in the club that is mapped to him.

ii. We can use the previous theorem that for every set U, $|2^U| > |U|$ to get many great results. For example, if U is the set of natural numbers $\mathcal{N}$, then we get that $|2^{\mathcal{N}}| > |\mathcal{N}|$ giving that the set $2^{\mathcal{N}}$ of subsets of $\mathcal{N}$ is uncountable. (We did not prove it, but $2^{\mathcal{N}}$ has the same cardinality as the reals $\mathcal{R}$.) As a second example, let U be the set of reals $\mathcal{R}$, then we get that $|2^{\mathcal{R}}| > |\mathcal{R}|$ giving that the set $2^{\mathcal{R}}$ of subsets of $\mathcal{R}$ is a bigger infinity than the number of reals. Prove by that there are a hierarchy of an infinite number of different sizes of infinity.

6. Computable and Describable Reals

(a) A real number x is said to be *computable* if there is a Java program that on input zero prints out the decimal representation of x from left to right. Note that at no point in time will all of the digits of x be printed out, but for each digit of x, there will be an eventual time at which this digit will be printed out. Use the words taught in class to prove in one sentence whether or not all real numbers computable.

(b) A real number x is said to be *describable* if it can be unambiguously denoted by a finite piece of English text. For example, $x = 2$ is described as "Two" and $x = \Pi$ as "The area of a circle of radius one." Use the words taught in class to prove in one sentence whether or not all real numbers describable.

(c) Prove that every computable real is also describable.

(d) Prove whether or not there a real number that can be described, but not computed? "Let x be the smallest real number that is not computable" is not a valid answer for the same real that "Let x be the smallest real number that is bigger than zero" is ill defined. Hint: Use a diagonalization proof.