York University
CSE 2001 – Unit 2 Models
Instructor: Jeff Edmonds

Read Jeff's notes. Read the book. Go to class. Ask lots of question. Study the slides. Work hard on solving these questions on your own. Talk to your friends about it. Talk to Jeff about it. Only after this should you read the posted solutions. Study the solutions. Understand the solutions. Memorize the solutions. The questions on the tests will be different. But the answers will be surprisingly close.

1. Quick

   (a) Loops: Do some computational problems require two loops or can all of them be accomplished by a single loop?

   (b) Suppose we want to design a TM to know whether a graph G is connected. What is the difference between G and $< G >$? When do we use $< G >$ and why is this done?

   (c) What is the difference between a computational problem that is a function vs one that is a language? If L is a language, what (if anything) do the following mean: $L(x)$, $x \in L$, $L = \emptyset$, $L = \{0,1\}^*$, L is decidable.

2. (Answer in Slides) TM Transitions (eg Shift Characters): In the slides we proved that a TM that is only able to write the characters $\{0, 1, b\}$ on the tape is as powerful as one that has a larger tape alphabet Σ. The simpler machine simulated the more complex one by grouping its cells into blocks of some fixed length so that each block contained a 0/1 code for the required character from Σ. Suppose this block size is four, input character 0 is encoded with 0000 and 1 with 0001. The simulation must start by inserting three zeros in front of every bit of the input. For example, on input $b1011bb\ldots$, the output of this first phase should be $\ldots bb0001000000010001bb\ldots$. Note that we don't actually care if the output is at the beginning of the tape as long as it is surrounded by blanks.

   (a) Writing down TM descriptions is hard. Instead, start by writing pseudo code using variables and loops. This is Jeff's level 4 abstraction of a TM. The only allowed actions are to read and write to the tape where the head is and to move the head to the left and right. I am also a big believer in *loop invariants*. This is a clear picture of what the tape looks like at the beginning of each iteration.
   Hint: Use the following Loop Invariant. For some $i \in [0, n]$, the first i bits of the input have been blanked out. These bits have been copied into a block past the input separated by a single blank with three zeros inserted before each. The head is on the b before

   (b) (Sorry: see the answer to the previous question before continuing.)
   Your next task is to prove that you know how to translate this code into a TM. You do not need to give the full translation. That would be far too tedious. Suppose a program had 13 lines of code and two variables $x, y \in \{1..5\}$, then what Pooh bear should write on the wall is the current line number that is being executed and the value of x and of y. Hence for each $k \in \{1..13\}, x', y' \in \{1..5\}$, let $q_{\langle line=k, x=x', y=y' \rangle}$ denote the state in which the TM is on line k, x has the value x', and y has the value y'. What are the states for the above program and how many of them are there? Give the general idea for how the transition function $\delta(q, c) = \langle q', c', direction \rangle$ is defined.

   (c) Consider line 2 of the code,
   i.e. "a = bit at head (i.e. the $i{+}1^{st}$ input bit), Write a blank, Move head right".

      i. This line of code is a level 3 abstraction of a TM, i.e. we interpret these blocks of transitions as lines of Java code. Explain what this code gets the TM to do.

      ii. Give a level 1 abstraction of this line, namely "Meaningful State Names". This involves giving each relevant entry of the table defining $\delta(q, c) = \langle q', c', direction \rangle$. The change from the level 0 abstraction is that we have given states meaningful names $q_{\langle line=39, x=3, y=0 \rangle}$.

iii. Give level 2 abstraction, namely "Meta Names and Transition Function". We talk about a block of states $q_{\langle line=39, x=x, y=y\rangle}$. for $x \in \{0,1,2,3,4\}$, and $y \in \{0,1,2,3,4\}$. We fill in blocks of the transition function table $\delta(q_{\langle line=39, x=x, y=y\rangle}, c) = \langle q_{\langle line=40, x=x, y=c\rangle}, c, right\rangle$.

(d) Do the same for lines 3 and 4, i.e. "Move head right until it is at a blank" and "Move head right one (i.e. into the output)".

(e) Do the same for line 9, i.e. "Write a".

(f) Bonus 10 marks if correct: The problem is to redo the previous problem, except the TM can only write 0 or 1. The tape starts with a two blanks before the input and an infinite number after it, but once a blank gets a 0/1 written in it, it will never be blank again. Hint: At first I was thinking that this could not be done, because the TM could not differentiate between whether what is written on the tape is the initial input or some markings to know that progress had been made. Then I saw how to do it. See if you can do it too.

3. (Answer in Slides) TM Transitions (eg Sort): We will design a TM which sorts an array of some n values each in the range $\{1, \ldots, d\}$. Each TM cell can contain one value from $\{-\infty, 1, \ldots, d, blank\}$.

algorithm InsertionSort(Tape of values)

***⟨pre−cond⟩*:**
The TM tape contains the n input valves in the cells "indexed" by $\{1, \ldots, n\}$.
Cell 0 contains $-\infty$ (so that it is already sorted).
Cell $n+1$ contains a blank (to indicate the end).
The head is on cell 1.

***⟨post−cond⟩*:**
The tape still contains the same n input values in the same n cells except that the valves been sorted.
The head is on the blank at cell $n+1$.

begin
01: allocate memory for a new variable p
02: let $p = -\infty$
03: loop

***⟨Measure−of−Progress 1⟩*:**
The tape still contains the same n input values except that some have been sorted. Denote by r the largest index so that valves in the cells indexed by $\{1, \ldots, r\}$ are sorted.

***⟨loop−invariant 1⟩*:**
Denote by i the cell index of the head. The values are sorted up to but maybe not including cell i, i.e. $i \in \{1, \ldots, r+1\}$. Also the TM has a variable p in which it remembers the value in the previous cell, i.e. the value in cell $i-1$.

***⟨Goal 1⟩*:**
To make progress by increasing r while maintaining this loop invariant.
All the discussions within this loop are relative to the tape values as they are here now.

04: loop

***⟨loop−invariant 2⟩*:**
The tape contents has not changed.
The loop invariant is LI1.
The change is that the head may have moved right.

***⟨Goal 2⟩*:**
To make progress by increasing i while maintaining this loop invariant.

05: let c denote the value at head
06: if(c is a blank) then
07: exit and return from entire sort program

08: else if($p \leq c$) then
09: $p = c$
10: move the head right
11: else
12: goto line 15
13: end if
14: end loop

15: forget the value in p and deallocate the memory
16: allocate memory for a new variable q
17: q = value at head

18: loop

⟨*loop−invariant* 3⟩:

The index r has been found, i.e. the first cell such that $Tape[r] > Tape[r+1]$.
The too small value that had been in cell $r+1$ has been saved in variable q.
The head is on cell i, for some $i \in \{1, \ldots, r+1\}$.
For each value $j \in \{i, \ldots, r\}$, the value that had been in cell j has been moved to cell $j+1$.
For example, the last sorted value, i.e. that in cell r, was shifted to cell $r+1$ where the smaller value had been
and the value in cell i, the current cell, has already been shifted over by one.
Note that we don't need the value in the current cell and hence this space can be used for something else.
Also, all of these shifted values are greater than the too small value that had been in cell $r+1$ and that is now saved in q.

⟨*Goal* 3⟩:

To make progress by decreasing i while maintaining this loop invariant.

19: write q into cell at head
20: forget the value in q and deallocate the memory
21: move head left
22: reallocate memory for variable p
23: let p = value at head
24: move head right
25: let q denote the value at head
(ie we know that this is value that had been in q, but we are not storing it)
26: if($p \leq q$) then goto line 3
27: allocate memory for q
28: let q = value at head
29: write p into cell
30: deallocate the memory for p
31: move head left
32: end loop
33: end loop
end algorithm

Your tasks:

(a) Translate this code into TM transitions.
Hint: Your task is to specify the start state and to fully define the transition function $\delta(q, c) = \langle q', c', direction \rangle$. Fill out big blocks of this table as follows:

- The first step is to choose some of the lines of code to be *check points* where the TM pauses and changes states. For each of these lines ℓ, you will need to define a block of state transitions specifying what the TM will do from states $q_{\langle line=\ell,\ldots \rangle}$.
  - Generally, there should be such a check point at the top of each loop. These take advantage of the loop invariant specified there.
  - The TM can keep running from a check point until it moves its head. At, the line ℓ' after this, the TM needs to use a transition $\delta(q_{\langle line=\ell',\ldots \rangle}, c)$ to read the character c that is at the TM's new head position.
    Warning: Because you do not have an loop invariant specified at this line, you may have to some extra thought asserting what is true at this point in the computation.
  - An optimal TM never needs to transition states unless it is moving its head. Sometimes, however, the TM is simpler if you put in additional check points. A TM transitions without moving the head by specifying "stay" instead of "right" or "left" for the head direction parameter, i.e. $\delta(q, c) = \langle q', c, stay \rangle$.
    Don't do this too often because we like the TM transitions to be neat and compact.
  - You might think that the first line of code needs a check point. Not so. The TM can start at any line before a value in the tape is used.
  - There needs to be a "check point" that the TM goes to when the computation exits. If you want the TM to halt when done, then this will be the TM's halt state. On the other hand, you might want the TM to go on to computing something else when the piece of code currently being considered is done. If the next code to be executed is assumed to be sequentially after this, then have a check point at the line ℓ after your code. If you are assuming that your code was a subroutine call, then the state of the TM might need to be remembering which line of code the function "return" statement returns the computation to, i.e. the TM will have to remember the information in the stack frames.

  List the line numbers ℓ at which the above code needs a check point.
- For each of the check points in the code, determine what you know is true at this point in the computation. If you have a loop invariant here, it will tell you. Otherwise, you have to make your own assertions.
- Give names to each of the state that the TM might be in when at this check point.
  - Recall that the name of the state $q_{\langle line=\ell, values\ known \rangle}$ must specify everything that the TM knows, i.e. everything written on Pooh bear's black board.
  - For example, suppose the assertion at code line ℓ states that the JAVA computation currently remembers the value variable p and q. The corresponding TM would be in state $q_{\langle line=\ell, p=p', q=q' \rangle}$.
  - Suppose that original range of p and q is $\{1, 2, 3, 4, 5\}$. However, the assertion goes on to state that the computation has learned $p \neq 2$ and $p < q$. The possible states then are:
    $\langle p', q' \rangle \in \{\langle 1, 2 \rangle, \langle 1, 3 \rangle, \langle 1, 4 \rangle, \langle 1, 5 \rangle, \langle 3, 4 \rangle, \langle 3, 5 \rangle, \langle 4, 5 \rangle\}$.
    Stated equivalently as:
    $\forall p', q' \in \{1, 2, 3, 4, 5\}$, where $p' \neq 2$ and $p' < q'$, we might be in state $q_{\langle line=\ell, p=p', q=q' \rangle}$.
- What the JAVA computation and the TM does next depends on the value c in the cell where the TM head is. The TM may have read this value before. As such, the ℓ assertion might have some previous knowledge about this value. For example, we might know that $p < c \leq q$.
- We will now fill in the block of the transition table with rows $\langle p', q' \rangle$ can columns c quantified by:
  $\forall p', q', c \in \{1, 2, 3, 4, 5\}$, where $p' \neq 2$ and $p' < c \leq q'$,
  we fill in table entry $\left\langle q_{\langle line=\ell, p=p', q=q' \rangle}, c \right\rangle$ with the actions taken by the TM.

- For each such $\langle q_{\langle line=\ell,p=p',q=q'\rangle}, c\rangle$, we determine what the corresponding TM will do by running in your mind the Java program from this check point until it reaches another check point.
  - As done above, name the state $q_{\langle line=\ell',p=p''\rangle}$ the TM will be in when reaching this next check point. Here ℓ' denotes index of the line code that the computation reaches and p'' is the new value for p. In this example, the computation forgot the value of q.
  - Let c' denote the value that is written into the cell where the head was during this block of computation. If no new value is written, then c' will simply be the value c that was there before.
  - Let *direction* denote the direction the head moves.
  - The transition then is:
    $\delta(q_{\langle line=\ell,p=p',q=q'\rangle}, c) = \langle q_{\langle line=\ell',p=p'',q=q''\rangle}, c', direction\rangle$.
  - If the computation between these two check points passes a fork, eg an "if" statement or the condition within "while" statement, then which case is followed might depend on the values $\langle q_{\langle line=\ell,p=p',q=q'\rangle}, c\rangle$. For example, suppose the computation forks based on whether the value of p is even or odd. Specify these parts of the transition table separately:
    $\forall p', q', c \in \{1,2,3,4,5\}$, where $p' \neq 2$ and $p' < c \leq q'$,
    if p' is even
    $\delta(q_{\langle line=\ell,p=p',q=q'\rangle}, c) = \langle q_{\langle line=\ell',p=p''\rangle}, c', direction\rangle$
    else
    $\delta(q_{\langle line=\ell,p=p',q=q'\rangle}, c) = \langle q_{\langle line=\ell'',q=q''\rangle}, c'', direction'\rangle$
- Determine the start state of the TM in this same way, i.e. run the JAVA computation from its beginning until it first reaches a check point and then name the state that the corresponding TM would be in.
- If the TM finds itself in a state that it did not expect, then we could have it write nice error messages. I tend to just put the TM into the panic state:
  $\forall p', q', c \in \{1,2,3,4,5\}$, where it is not the case that $[p' = 2$ and $p' < c \leq q']$,
  $\delta(q_{\langle line=\ell,p=p',q=q'\rangle}, c) = \langle q_{\langle line=panic\rangle}, -infty, stay\rangle$.
- Be sure that all these blocks of the table that you are filling out are a disjoint union of the entire table.

Hint: My answer does this for about a dozen such blocks.

(b) How many states does your TM have?

(c) Lines 15, 16, 19, 20, 22, 27, and 30 of the above Java program do a lot of allocating and deallocating of memory. If we do not do this, what changes?

(d) What is the big-Oh running time of your TM (in terms of TM steps)?
Does in depend on d? Why or why not?
You may know that the advertised running time of Insertion Sort is $O(n^2)$. This TM seems to do a lot in one time step, but also must move its head back and forth a lot. So, is its running time more or less than $O(n^2)$?
Analyze and explain.

4. (Answer in Slides) TM computing $J(I)$

(a) The input consists two blocks each consisting of exactly a million 0/1 characters. The first is the binary description of a JAVA program J, the second of an input I. Describe, if possible, a simple TM for computing whether $J(I) = 1$, meaning that program J on this input halts and accepts or not, $J(I) = 0$. Focus on giving a meaningful name to each of the states of your machine. Describe the complete $\delta(q, c) = \langle q', c', direction\rangle$ function in all of its details.

(b) Redo the question when J is arbitrarily long, but I is still of this fixed length. Remember that lots of problems can be solved that have arbitrarily long inputs so if you want to say that this one is uncomputable, don't give this as your reason.

5. (Answer in Slides) Recall that a Universal TM is a TM $M_{universal}$, which when given input $\langle$"M", $I\rangle$, simulates TM M on input I. You are to give a sketch of how such a $M_{universal}$ would be built.
To make it easier, lets assume that M only has one tape and one head.
Hint: Because $M_{universal}$ is one fixed TM, it must have one fixed set of states $Q_{universal}$ and one fixed tape alphabet $\Sigma_{universal}$ (i.e. the set of characters c that it is allowed to write in each tape call.)
Hint: Fix $\Sigma_{universal} = \{\ '0',\ '1',\ '\langle\ ',\ ',',\ '\rangle'\ \}$.
Hint: Let $M_{universal}$ have three tapes each with its own head.
When $M_{universal}$'s computation begins, the description of M will appear on $M_{universal}$'s first tape and I on its second.

   (a) We must assume that the TM M described in $M_{universal}$'s input has an arbitrarily large set of states and an arbitrarily large tape alphabet Σ_M.
   Explain how you would describe M on $M_{universal}$'s first tape. Remember this description must be encoded using characters from $\Sigma_{universal} = \{\ '0',\ '1',\ '\langle\ ',\ ',',\ '\rangle'\ \}$.

   (b) At the beginning of $M_{universal}$'s simulation of the ${t_M}^{th}$ time step of M's computation, $M_{universal}$'s first tape will still contain the description of M. What would be best to put on $M_{universal}$'s second and third tape? Remember what you have written there must be encoded using characters from $\Sigma_{universal} = \{\ '0',\ '1',\ '\langle\ ',\ ',',\ '\rangle'\ \}$.
   Also where will $M_{universal}$ three tape heads be.

   (c) We say that M can compute in one time step any function of "what it knows", where "what it knows" is its current state and the character that its head currently on. More formally, if M is in state q and its head is seeing the character c, then $\delta_M(q, c) = \langle q', c', direction\rangle$ specifies M's next state and next actions.
   Is it possible that $M_{universal}$ "knows" q, or c, or for that matter δ_M?

   (d) Quickly sketch the actions taken by $M_{universal}$'s simulation of the ${t_M}^{th}$ time step of M's computation.

   (e) Let $T(|I|)$ denote the running time of M on I and let $|$"M"$|$ denote length of the description of M stored on $M_{universal}$'s first tape.
   What is the running time needed by $M_{universal}$ to simulate $M(I)$?

   (f) It is not fair that $M_{universal}$ is allowed three tapes but M only one. Suppose instead, $M_{universal}$ was only allowed one tape.

   Remember that a one tape TM $M_{universal}$ is able to simulate our three tape $M_{universal}$ by interweaving the cells of the three tapes on one tape or by having a larger tape alphabet $\Sigma_{universal}$ to include characters like $'0\langle\ 1'$ to signify that $'0'$ is on its first tape $'\langle\ '$ on its second and $'1'$ on its third.

   The challenge is that the one tape version of $M_{universal}$ has one not three heads.

   Show how the straight forward implementation requires $[\mathcal{O}(|$"M"$|) + \mathcal{O}(T(|I|))] \times T(|I|)$ time.

   Show how this can be improved to only $\mathcal{O}(|$"M"$|) \times T(|I|)$ time with the smart trick of moving the "M" and the "q".

6. (Solution in Slides)

   Think carefully about the definitions of Turing Machines. For every state q that the machine might currently be in and character c that might be written in the cell of the tape pointed to by the head, we define $\delta(q, c) = \langle q', c', direction\rangle$ where q' is the next state, c' is the character to write, and *direction* is the direction in which to move the head one cell.

   For each of these three specified changes to the TM model, explain the effect it would have on the reasonableness and on the computational power of the model. Do one of:
   - A: Either argue that the set of problems computable does not change because the new and the old versions of TM can simulate each other
   - B: or show that the new model is weaker by giving a computable problem that no longer can be computed

- C: or show that the new model is stronger by giving an easy algorithm for an arbitrary uncomputable problem (for example the halting problem).

(a) Instead of requiring the characters c that are written in each cell of the tape to be from a set of a fixed size (eg binary or ascii), allow the c and the c' in the above definition $\delta(q, c) = \langle q', c', direction \rangle$ to be arbitrarily large integers.

(b) Change the model so that the TM "knows" the location of the head. More formally, $\delta(q, i, c) = \langle q', c', direction \rangle$ also depends on the index i of the cell that head is currently at.

(c) Change the model so that the TM no longer has a head but can specify the index of a cell of the tape to read and to write to. The TM still has its input stored in the first n cells of the input and is only allowed a fixed/constant/finite number of states and alphabet size. Formally, $\delta(q, c) = \langle q', c', i, i' \rangle$ where c is the last value read, c' will be written to cell indexed by i and the next c will be read from that indexed by i'.

7. Eric Ruppert had a contest to make as a TM with only 6 states that starting with string 11111100000 runs for as long as possible. I gave him a solution with only 5 states that ran for 3,983,884 time steps.

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My idea is think of the contents of the tape in binary and to count
down and to extend the length of the used tape by one each decrement.
However, we flip the role of zeros and ones.  And we flip the order of
the bits from left to right.

Our initial tape is to contain #11111100000.  (Here # indicates the
start of the tape)
Changing zeros to ones and ones to zeros
and flipping the order of the bits from left to right gives
 11111000000#, which is 1984 in binary.
Each meta step will decrement this number by one and add a leading
zero.
The loop invariant is that after the i^th meta step, the number in
binary will be 1984-i with i leading zeros.
For example,
i=0: _________11111000000#  = 1984  with 0 leading zeroes
i=1: ________011110111111#  = 1983  with 1 leading zeroes
i=2: _______0011110111110#  = 1982  with 2 leading zeroes
i=3: ______00011110111101#  = 1981  with 3 leading zeroes
i=4: _____000011110111100#  = 1980  with 4 leading zeroes
i=5: ____0000011110111011#  = 1979  with 5 leading zeroes

Changing ones back to zeros and zeros back to ones
and flipping the order of the bits back again from right to left then
gives
the actual tape content to be
i=0: #11111100000_________
i=1: #000000100001________
i=2: #1000001000011_______
i=3: #10000010000111______
i=4: #110000100001111_____
i=5: #0010001000011111____

Let n = 1984 be our initial value.
Let m = 11 be the initial number of bits.
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The i^th meta step   (going from i-1 to i)
will change the integer from n-i+1 to n-i
and change the number of bits from m+i-1 to m+i.
To do this, the algorithm must move the from left to the right and
back to the left.
This takes 2(m+i) time.
The last pass of the TM will just go once till it finds the blank.
Hence, the total time is
  [sum_{i = 1..n}  2(m+i)] +  m+n
  = 2mn + [sum_{i = 1..n}  2i] + m+n
  = 2mn + n(n+1) + m+n
  = 2(11)(1984) + (1984)(1985) + 11 + 1984
  = 3,983,883

To make it easier to understand,
I am first going to write the code with zeros and ones switched
and left and right flipped.

Suppose the current state is
  i=4: _____000011110111100#  = 1980  with 4 leading zeroes
  with the head on the right most bit, i.e., the low order bit  (here a 0)
  in state s0
and we must change it to
  i=5: ____0000011110111011#  = 1979  with 5 leading zeroes
  with the head on the right most bit, i.e., the low order bit  (here a 1)
  in state s0

Psuedo Code:
Move left changing zeros to ones until you reach the rightmost 1.
Change this one to zero and go one more step left.
The algorithm then moves the head all the way to the left to the first
blank and changes it to a 0.
The algorithm then moves the head back all the way from left to right.
This reestablishes the loop invariant.
The code terminates when the integer on the tape is all zero.

Translate this into TM transitions.
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