EECS 1090 – Test 1

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1. (20 marks) Fill out the table with all of the rules.

	Proof Techniques/Lemmas			
	Using		Proving	
	From:	Conclude	From:	Conclude
And $\wedge$:	Separating And		Eval/Build/Simplify $\wedge$	
	$\alpha \wedge \beta$	α & β	α & β	$\alpha \wedge \beta$
			$\neg\alpha$	$\neg(\alpha \wedge \beta)$
			α	$\alpha \wedge \beta$ iff β
Or $\vee$:	Selecting Or		Eval/Build/Simplify $\vee$	
	$\alpha \vee \beta$ & $\neg\alpha$	β	α	$\alpha \vee \beta$
			$\neg\alpha$ & $\neg\beta$	$\neg(\alpha \vee \beta)$
			$\neg\alpha$	$\alpha \vee \beta$ iff β
	Cases		Excluded Middle	
	$\alpha \vee \beta$, $\alpha \rightarrow \gamma$, & $\beta \rightarrow \gamma$	γ		$(\alpha \vee \alpha)$ & $\neg(\alpha \wedge \alpha)$
Implies $\rightarrow$:	Modus Ponens		Deduction	
	α & $\alpha \rightarrow \beta$	β	Assume α, prove β	$\alpha \rightarrow \beta$
	Cases		Eval/Build/Simplify $\rightarrow$	
	$\alpha \vee \beta$, $\alpha \rightarrow \gamma$, & $\beta \rightarrow \gamma$	γ	$\neg\alpha$	$\alpha \rightarrow \beta$
			β	$\alpha \rightarrow \beta$
			α & $\neg\beta$	$\neg(\alpha \rightarrow \beta)$
			α	$\alpha \rightarrow \beta$ iff β
			$\neg\beta$	$\alpha \rightarrow \beta$ iff $\neg\alpha$
	Equivalence		Contrapositive	
	$\alpha \rightarrow \beta$ & $\beta \rightarrow \alpha$	α iff β	$\alpha \rightarrow \beta$ iff $\neg\beta \rightarrow \neg\alpha$ iff $\neg\alpha \vee \beta$	
	Transitivity		De Morgan's Law	
	$\alpha \rightarrow \beta$ & $\beta \rightarrow \gamma$	$\alpha \rightarrow \gamma$	$\neg(\alpha \wedge \beta)$ iff $\neg\alpha \vee \beta$	

2. (13 marks) Find all possible assignments of the variables that makes the following expression true/satisfied.
Explain all of the steps in your search for the assignment and in proving that this assignment works.
Use Purple table reasoning, not a table.
Hint: Start with proof/search by cases with the $p \vee q$, then see how you can force the values of other variables.

$$[p \vee q] \wedge [p \rightarrow s] \wedge [\neg p \vee \neg s] \wedge [t \rightarrow \neg q] \wedge [u \vee t] \wedge [u \oplus v] \wedge [w \rightarrow \neg w] \wedge [y \rightarrow (x \wedge \neg x)].$$

How many different satisfying assignments are there?

- Answer: The AND between the clauses means that each of them needs to be true.

  Clause $p \vee q$: We don't which of these is true, so as hinted we will do search by cases. Let's first guess that p is true.

  Clause $p \rightarrow s$: p true and Modus ponens with this clause forces $s = T$.

  Clause $\neg p \vee \neg s$: We set both p and s true, making this term false. This is a contradiction. Hence, we need to back up and set p to be false.

  Clauses $p \rightarrow s \wedge \neg p \vee \neg s$: With p false, both of these terms are automatically true. This allows s to be true or false.

  Clause $p \vee q$: Because it did not work to set p to be true, we are now forced to set q true.

  Clause $t \rightarrow \neg q$: q true, contra positive, and modus ponens (modus tollens) forces $t = F$.

  Clause $u \vee t$: t false and selective or forces $u = T$.

  Clause $u \oplus v$: u true and parity forces $v = F$.

  Clause $w \rightarrow \neg w$: Let's do proof by cases.
  If $w = T$, then $w \rightarrow \neg w$ gives that $w = F$. This is a contradiction.
  If $w = F$, then $w \rightarrow \neg w$ is automatically true.

Hence, w is forced to be false.
Another technique is to translate $\alpha \to \beta$ into $\neg\alpha \vee \beta$ and $w \to \neg w$ into $\neg w \vee \neg w$. This forces $w = F$.

Clause $y \to (x \wedge \neg x)$: $x \wedge \neg x$ is false by exclusive middle no matter what x is. Contra positive, and modus ponens (modus tollens) forces $y = F$.

In conclusion, $p = F$, $q = T$, $s =?$, $t = F$, $u = T$, $v = F$, $q = T$, $w = F$, $x =?$, and $y = F$, makes each clause true.
Because the values of s and x are not forced, there are $2 \times 2 = 4$ satisfying assignments.

3. Nicer Father

(a) (8 marks) It is not true that
$[(\alpha \to \gamma)$ *and* $(\beta \to \gamma)]$ iff $[(\alpha$ *and* $\beta) \to \gamma]$.
Thinking of $\to$ as causality, argue in English as you would to someone who does not know logic why this statement is false.

- Answer: The left hand side only needs one of α or β to be true to ensure that γ is true. On the other hand, the right hand side only needs both α and β to be true to ensure that γ is true.

(Deleted) The purple table says how $\alpha \to \gamma$ is equivalent to a statement with an *or* or an *and*. Do this with the following and continue to rearrange them until it is clear which are equivalent and which are different. Hint: Try factoring out something using reverse distributive law.

i. $(\alpha \to \gamma)$ *and* $(\beta \to \gamma)$
   - Answer: $\equiv (\neg\alpha \text{ or } \gamma) \text{ and } (\neg\beta \text{ or } \gamma)$
     $\equiv (\neg\alpha \text{ and } \neg\beta) \text{ or } \gamma$
     Factoring out γ using reverse distributive law.

ii. $(\alpha \text{ and } \beta) \to \gamma$
   - Answer: $\equiv \neg(\alpha \text{ and } \beta) \text{ or } \gamma$
     $\equiv \neg\alpha \text{ or } \neg\beta \text{ or } \gamma$

iii. $(\alpha \text{ or } \beta) \to \gamma$
   - Answer: $\equiv \neg(\alpha \text{ or } \beta) \text{ or } \gamma$
     $\equiv (\neg\alpha \text{ and } \neg\beta) \text{ or } \gamma$

iv. Which the above three are equivalent and which are different?
   - Answer: The first and third are equivalent. The middle is not.

(b) Use the purple table to prove the following two statements.
Use deduction. Do NOT convert the $\to$ into *and* or *or*.

i. (13 marks) $[(\alpha \to \gamma) \text{ and } (\beta \to \gamma)] \to [(\alpha \text{ or } \beta) \to \gamma]$.

   - Answer:

1)	Deduction Goal: $[(\alpha \to \gamma) \text{ and } (\beta \to \gamma)] \to [(\alpha \text{ or } \beta) \to \gamma]$	
2)	$(\alpha \to \gamma) \text{ and } (\beta \to \gamma)$	Assumption/Premise
3)	$\alpha \to \gamma$	Separating And 2
4)	$\beta \to \gamma$	Separating And 2
5)	Deduction Goal: $(\alpha \text{ or } \beta) \to \gamma$	
6)	$\alpha \text{ or } \beta$	Assumption/Premise
7)	γ	Cases 3,4, & 6
8)	$(\alpha \text{ or } \beta) \to \gamma$	Conclude deduction.
9)	$[(\alpha \to \gamma) \text{ and } (\beta \to \gamma)] \to [(\alpha \text{ or } \beta) \to \gamma]$	Conclude deduction.

ii. (13 marks) $[(\alpha \rightarrow \gamma) \text{ } and \text{ } (\beta \rightarrow \gamma)] \leftarrow [(\alpha \text{ } or \text{ } \beta) \rightarrow \gamma]$.

- Answer:

1)	Deduction Goal: $[(\alpha \rightarrow \gamma) \text{ } and \text{ } (\beta \rightarrow \gamma)] \leftarrow [(\alpha \text{ } or \text{ } \beta) \rightarrow \gamma]$	
2)	$(\alpha \text{ } or \text{ } \beta) \rightarrow \gamma$	Assumption/Premise
3)	Deduction Goal: $\alpha \rightarrow \gamma$	
4)	α	Assumption/Premise
5)	$\alpha \text{ } or \text{ } \beta$	Building/Eval Or
6)	γ	Modus Ponens 2 & 5
7)	$\alpha \rightarrow \gamma$	Conclude deduction.
8)	$\beta \rightarrow \gamma$	Similar to 3-7
9)	$(\alpha \rightarrow \gamma) \text{ } and \text{ } (\beta \rightarrow \gamma)$	Building And
9)	$[(\alpha \rightarrow \gamma) \text{ } and \text{ } (\beta \rightarrow \gamma)] \leftarrow [(\alpha \text{ } or \text{ } \beta) \rightarrow \gamma]$	Conclude deduction.

(c) (7 marks) Jeff told a story about a grumpy father, doing well at school, following rules, and being loved. Change the contract of the father to be $[(\alpha \rightarrow \gamma) \text{ } and \text{ } (\beta \rightarrow \gamma)]$. How is this father different than in Jeff's story.

- Answer: In Jeff's story, the father did not have clear requirements for his daughter to be loved. Doing well at school might have been sufficient. Following rules might have been sufficient. It just was not clear which. The changed father is clear that one or the other is sufficient.

4. (13 marks) The game *Ping* has two rounds. Player-A goes first. Let m_1^A denote his first move. Player-B goes next. Let m_1^B denote his move. Then player-A goes m_2^A and player-B goes m_2^B. The relation $AWins(m_1^A, m_1^B, m_2^A, m_2^B)$ is true iff player-A wins with these moves.

Use universal and existential quantifiers to express the fact that player-A has a strategy in which she wins no matter what player-B does. Use $m_1^A, m_1^B, m_2^A, m_2^B$ as variables. Explain.

- Answer: $\exists m_1^A \forall m_1^B \exists m_2^A \forall m_2^B \, AWins(m_1^A, m_1^B, m_2^A, m_2^B)$. It states that player-A's strategy has a first move m_1^A, such that no matter what player-B's move m_1^B is, player-A's has a second move m_2^A, such that no matter what player-B's move m_2^B is, player-A wins.

(Deleted)

(a) What steps are required in the Prover/Adversary technique to prove this statement?

- Answer: The proof follows the game.
  Player-A, as the prover, specifies her first move m_1^A. Let Player-B's move m_1^B be arbitrary.
  Player-A's specifies her second move m_2^A. Let Player-B's second move m_2^B be arbitrary.
  Prove $AWins(m_1^A, m_1^B, m_2^A, m_2^B)$

(b) What is the negation of the above statement in standard form? Explain what it means.

- Answer: The negation is $\forall m_1^A \exists m_1^B \forall m_2^A \exists m_2^B \neg AWins(m_1^A, m_1^B, m_2^A, m_2^B)$. It states that player-A does not have a winning strategy. Given someone always does, it follows that player-B has a strategy in which he wins no matter what player-A does, namely no matter what player-A's first move m_1^A is, player-B has a first move m_1^B, such that no matter what player-A's second move m_2^A is, player-B has a second move m_2^B, such that player-A loses.

(c) What steps are required in the Prover/Adversary technique to prove this negated statement?

- Answer: The proof follows the game.
  Let Player-A's move m_1^A be arbitrary.
  Player-B, as the prover, specifies his first move m_1^B. Let Player-A's second move m_2^A be

arbitrary.
Player-B specifies his second move m_2^B.
Prove $\neg AWins(m_1^A, m_1^B, m_2^A, m_2^B)$

5. (13 marks) We say that the sequence $f = f(1), f(2), f(3), \ldots$ converges if
$\exists c\ \forall \epsilon > 0\ \exists n_0\ \forall n \geq n_0\ |f(n) - c| \leq \epsilon$.
Play the Jeff's game in order to prove that the sequence $f = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots$, i.e., $f(i) = \frac{1}{i}$ converges.

   - Answer: Let $c = 0$. Let ϵ be arbitrary. Let $n_0 = \frac{1}{\epsilon}$. Let $n \geq n_0$ be arbitrary. Because $n \geq n_0 = \frac{1}{\epsilon}$, we have that $|f(n) - c| = |\frac{1}{n} - 0| = \frac{1}{n} \leq \epsilon$.